DIVIDENDS, EARNINGS, AND STOCK PRICES

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THE three possible hypotheses with respect to what an investor pays for when he acquires a share of common stock are that he is buying (1) both the dividends and the earnings, (2) the dividends, and (3) the earnings. It may be argued that most commonly he is buying the price at some future date, but if the future price will be related to the expected dividends and/or earnings on that date, we need not go beyond the three hypotheses stated. This paper will critically evaluate the hypotheses by deriving the relation among the variables that follows from each hypothesis and then testing the theories with cross-section sample data. That is, price, dividend, and earnings data for a sample of corporations as of a point in time will be used to test the relation among the variables predicted by each hypothesis.

The variation in price among common stocks is of considerable interest for the discovery of profitable investment opportunities, for the guidance of corporate financial policy, and for the understanding of the psychology of investment behavior.[1] Although one would expect that this interest would find expression in cross-section statistical studies, a search of the literature is unrewarding.

Cross-section studies of a sort are used extensively by security analysts to arrive at buy and sell recommendations. The values of certain attributes such as the dividend yield, growth in sales, and management ability are obtained and compared for two or more stocks. Then, by some weighting process, a conclusion is reached from this information that a stock is or is not an attractive buy at its current price.[2] Graham and Dodd go so far as to state that stock prices should bear a specified relation to earnings and dividends, but they neither present nor cite data to support the generalization.[3] The distinguished theoretical book on investment value by J. B. Williams contains several chapters devoted to the application of the theory, but his empirical work is in the tradition of the investment analyst's approach.[4] The only study along the lines suggested here that is known to the writer is a recent one on bank stocks by David Durand.[5]

In contrast with the dearth of published studies the writer has encountered a number of unpublished cross-section regressions of stock prices on dividends, earnings, and sometimes other variables. In these the correlations were high, but the values of the regression coefficients and their variation among samples (different industries or different years) made the economic significance of the results so questionable that the investigators were persuaded to abandon their studies. There is reason to believe that the unsatisfactory nature of the findings is due in large measure to the inadequacy of the theory employed in interpreting the model, and it is hoped that this paper will contribute to a more effective use of cross-section stock price studies by presenting what might be called the elementary theory of the variation in stock prices with dividends and earnings.

Before proceeding, it may be noted that there have been some time series studies of the variation in stock prices with dividends and other variables. The focus of these studies has been the relation between the stock market and the business cycle[6] and the discovery of profitable

* The research for this paper was supported by the Sloan Research Fund of the School of Industrial Management at Massachusetts Institute of Technology. The author has benefited from the advice of Professors Edwin Kuh, Eli Shapiro, and Gregory Chow. The computations were done in part at the M.I.T. Computation Center.

[1] Assume that the hypothesis stock price, $P = f(x_1, x_2, \ldots)$, is stated so that it can be tested, and it is found to do a good job of explaining the variation in price among stocks. The model and its coefficients thereby shed light on what investors consider and the weight they give these variables in buying common stocks. This information is valuable to corporations insofar as the prices of their stocks influence their financial plans. It is also true that a stock selling at a price above or below that predicted by the model deserves special consideration by investors.

[2] Illustrations of this method of analysis may be found in texts on investment analysis such as: Graham and Dodd, *Security Analysis*, 3rd ed. (New York, 1951); and Dowrie and Fuller, *Investments* (New York, 1941).

[3] Graham and Dodd, op. cit., 454 ff.

[4] *The Theory of Investment Value* (Cambridge, 1938).

[5] *Bank Stock Prices and the Bank Capital Problem*, Occasional Paper 54, National Bureau of Economic Research (New York, 1957).

[6] J. Tinbergen, "The Dynamics of Share-Price Formation," this REVIEW, XXI (November 1939), 153–60; and Paul G. Darling, "A Surrogative Measure of Business Con-

investment opportunities.[7] They have not been concerned with explaining the variation in price among stocks, and it is questionable whether such data can be effectively used for this purpose. Auto-correlation in the time series would impair the significance of the regression coefficients for many of the variables. Possibly even more important, the use of time series assumes that the coefficient of a variable is constant over time but different among stocks. The exact opposite is assumed in any attempt to explain preference among investment opportunities.

The Sample

To test each of the theories, price, dividend, and earnings data were obtained for four industries and two years, so that there are eight samples in all. The years chosen were 1951 and 1954, and the industries and number of corporations for each industry are Chemicals, 32; Foods, 52; Steel, 34; and Machine Tools, 46.

Including only those corporations which conformed to a narrow definition of the industries mentioned did not provide samples of adequate size. Therefore, certain fringe classifications were included in each category. For instance, Chemicals includes pharmaceutical manufacturers, and Steel includes forging manufacturers and certain other fabricators of steel as well as the basic steel producers. In general, while the corporations included in each sample can be considered to come under the label, there is considerable variation among them in such attributes as size, profitability, structure of the markets in which they buy and sell, and investor status.[8]

The use of eight samples rather than one provides a more rigorous test of the hypotheses. The industry and year selection of the data has the further advantage of allowing the use of *a priori* economic knowledge in evaluating the regression statistics. For instance, if the dividend coefficient is considered an estimate of the rate of profit, we want to know whether the estimate is reasonable on grounds broader than statistical significance. Good preferred stocks sold in these years at dividend yields of four to five per cent, and companies acquired in mergers were purchased for about five times their earnings before income taxes. Therefore, we would expect the rate of profit on common stocks to fall between four and ten per cent and the coefficient in question to fall between ten and twenty-five. Further, we would expect a particular rank in the coefficients. Corporations in the chemical industry are considered to have the advantages of size, growth, and stability; foods represent an industry that is considered stable; steels represent an industry with large corporations which are considered vulnerable to cyclical fluctuations; and machine tools represent an industry of comparatively small corporations which are also vulnerable to the business cycle. Accordingly, one might expect the rate of profit to vary among the industries in the order just given. Further, 1951 was a year of war profits with the outlook for the future somewhat uncertain. By contrast, while there was some talk of recession in 1954, there was little evidence that the high level of income extending back a number of years would fall sharply in the near future. Accordingly, one might expect that the coefficients would differ in a predictable manner between the two years.

Dividends and Earnings

Given the task of explaining the variation in price among common stocks, the investigator may observe that stockholders are interested in both dividend and income per share and derive immediately from this observation the model:

$$P = a_0 + a_1 D + a_2 Y \tag{1}$$

where P = the year-end price, D = the year's dividend, and Y = the year's income. The equation may be considered of interest solely for the multiple correlation between the actual and predicted price, in which case no meaning can be given to the regression coefficients. Alternatively, the equation may be read to mean that the coefficients a_1 and a_2 represent the value the market places on dividends and earnings respectively, a possible objective being the

fidence and Its Relation to Stock Prices," *Journal of Finance*, x (December 1955), 442–58.

[7] The outstanding example of this is *The Value Line Investment Survey*. In addition, numerous articles in the *Analysts Journal* and the *Journal of Finance* analyze the change over time of price with other variables. A paper of some interest is D. Harkavy, "The Relation Between Retained Earnings and Common Stock Prices for Large, Listed Corporations," *Journal of Finance*, VIII (September 1953), 183–97.

[8] A list of the corporations and a description of how they were selected may be obtained from the writer on request.

measurement of the relative importance of the two variables. However, a share of stock like any other asset is purchased for the expected future income it provides. This income may be the dividend or it may be the earnings per share, but it cannot be both. The model is therefore conceptually weak.

The unfortunate consequence of this pragmatic approach to the measurement of the variation in stock prices with dividend and earnings is illustrated by the data of Table 1. The dividend coefficient for chemicals in 1951 is negative and machine tools has the highest coefficient. Between 1951 and 1954 the chemicals coefficient changes from approximately zero to 25. Many of the dividend coefficients are materially below ten, and in 1954 the highest coefficient is five times the lowest. The income coefficients, with the exception of chemicals in 1951, are extraordinarily low as measures of the price the market is willing to pay for earnings.

TABLE 1. — MODEL I, REGRESSION OF PRICE ON DIVIDEND AND INCOME

Sample	Constant term	Coefficient and standard error of		Multiple correlation
		D	Y	
1951 – Chemicals	−7.0	−.8	16.7	.93
		(5.2)	(3.1)	
Foods	.1	7.0	5.5	.90
		(1.5)	(.9)	
Steels	5.5	6.6	2.0	.86
		(1.8)	(.6)	
Machine tools	2.4	12.0	.8	.90
		(1.2)	(.5)	
1954 – Chemicals	−3.0	25.7	.3	.92
		(5.2)	(3.3)	
Food	−.4	10.4	5.6	.91
		(2.2)	(1.0)	
Steels	8.7	8.4	2.0	.94
		(1.7)	(.8)	
Machine tools	6.3	5.5	4.1	.89
		(1.4)	(.6)	

Machine tools in 1951 and chemicals in 1954 have income coefficients that are not significantly different from zero, and three of the other coefficients are materially below five. Armed only with the theory just stated, it would be most difficult to infer from the data the existence of a logical structure in the pricing of common stocks.

The Dividend Hypothesis

The hypothesis that the investor buys the dividend when he aquires a share of stock seems intuitively plausible because the dividend is literally the payment stream that he expects to receive. In implementing the hypothesis it must be recognized that the stockholder is interested in the entire sequence of dividend payments that he may expect and not merely the current value. For the purpose of arriving at an operational model we may represent this infinite sequence by two quantities, one the current dividend and the other a measure of the expected growth in the dividend.

Among the events which will lead to an increase in a corporation's dividend are: successful trading on its equity, an increase in its return on investment, and selling additional common stock when the rate of profit the corporation can earn is above the rate at which its stock is selling. However, there is no doubt that the most important and predictable cause of growth in a corporation's dividend is retained earnings. For those interested in a more rigorous argument it has been shown that if a corporation is expected to earn a return r on investment and retain a fraction b of its income, the corporation's dividend can be expected to grow at the rate br.[9] If the investment or book value per share of common stock is B, then

$$br = \left(\frac{Y-D}{Y}\right)\left(\frac{Y}{B}\right) = \frac{Y-D}{B}. \qquad (2)$$

Investors are interested in growth and not rate of growth, since a high rate of growth starting with a low initial value will pay off in the heavily discounted distant future, and it will not be as attractive as a lower rate of growth starting from a higher initial value. Therefore, in a model where price and dividend are absolute quantities, it is likely that retained earnings per share without deflation by book value is a better measure of growth than the rate of growth.

The previous discussion has provided the economic rationale for using the equation

$$P = a_0 + a_1 D + a_2 (Y-D) \qquad (3)$$

[9] The argument is developed more fully in M. J. Gordon and Eli Shapiro, "Capital Equipment Analysis: The Required Rate of Profit," *Management Science*, III (October 1956), 102–10.

to represent the hypothesis that the investor buys the dividend when he acquires a share of stock. The reciprocal of the dividend coefficient may be looked on as an estimate of the rate of profit the market requires on common stocks without growth, and the retained earnings coefficient is the estimate of what the market is willing to pay for growth.

Table 2 presents the eight sample estimates of the model's coefficients. The 1951 dividend coefficients are considerably superior to those of Model I under the criteria stated earlier for their absolute and relative values. Only the machine tools coefficient appears comparatively high. The 1954 coefficients vary among the industries as expected and they fall within the expected range. The spread in the coefficients is only one-half the range of those in Model I, but it still seems quite large. In particular one might wonder at the high chemicals-1954 coefficient, the low steels-1951 and machine tools-1954 values, and the strong inverse correlation between the coefficients and the constant terms.

Turning now to the retained earnings coefficients, what would we expect of them? Since they represent the price the market is willing to pay for growth in the dividend, with retained earnings serving as an index of growth, the only statement with respect to their values that follows from the theory is that they should be positive. It may be thought nonetheless that their values seem low, and the absence of statistical significance at the five per cent level for two coefficients, machine tools-1951 and chemicals-1954, is particularly disturbing. The really surprising result is the negative chemicals coefficients for 1954. On the other hand there is some *a priori* credibility in the findings. Growth is most uncertain and it becomes quantitatively important by comparison with the current dividend in the distant future. Also, apart from the 1954 chemicals there is a rough correspondence between the rank of the coefficients and notions as to the comparative stability of earnings among the industries.

The reader may have noted (1) the multiple correlation coefficients in Tables 1 and 2 are the same for each industry year, (2) the earnings and retained earnings coefficients, a_2 and α_2 are the same, and (3) the dividend coefficient $\alpha_1 = a_1 + a_2$. On the first point, in both equations price is a linear function of the same variables, so that they both yield the same correlation coefficients. The earnings and retained earnings coefficients are the same, since the change in earnings is the same as the change in retained earnings when the dividend is held constant. The difference in the dividend coefficients is due to the fact that in equation (1) the increase in dividend involves a corresponding reduction in retained earnings, whereas in equation (3) retained earnings is held constant.

TABLE 2. — MODEL II, REGRESSION OF PRICE ON DIVIDEND AND RETAINED EARNINGS

Sample	Constant term	Coefficient and standard error of D	$Y - D$	Multiple correlation
1951 – Chemicals	−7.0	15.9	16.7	.93
		(2.7)	(3.1)	
Foods	.1	12.5	5.5	.90
		(1.1)	(.9)	
Steels	5.5	8.6	2.0	.86
		(1.5)	(.6)	
Machine tools	2.4	12.8	.8	.90
		(1.0)	(.5)	
1954 – Chemicals	−3.0	30.0	.3	.92
		(2.6)	(3.3)	
Foods	−.4	15.9	5.6	.91
		(1.5)	(1.0)	
Steels	8.7	10.4	2.0	.94
		(1.4)	(.8)	
Machine tools	6.3	9.6	4.1	.89
		(1.2)	(.6)	

The dividend hypothesis provides a more reasonable interpretation of equation (1) than the interpretation given in the previous section. If growth is valued highly, an increase in the dividend with a corresponding reduction in retained earnings will not increase the value of a share as much as when a low value is placed on growth. There is some tendency for the a_1 coefficients to vary among industries accordingly. Another point to be noted is that the standard error of α_1 is below that for a_1. This combined with the higher values of the former coefficients means that the change in price with the dividend can be predicted with much greater accuracy when retained earnings are held constant than when the increase comes out of retained earnings.

The Earnings Hypothesis

The third hypothesis is that the investor buys the income per share when he acquires a share of stock. The rationale is that regardless of whether they are distributed to him the stockholder has an ownership right in the earnings per share. He receives the dividend in cash and the retained earnings in a rise in the share's value, and if he wants additional cash he can always sell a fraction of his equity. In short, the corporate entity is a legal fiction that is not material with respect to his rights in the corporation or the value he places on them.[10] One can argue further that the different tax treatment of dividends and capital gains creates a stockholder preference for retained earnings.

The hypothesis may be tested by reference to the data of Table 2. If the investor is indifferent to the fraction of earnings distributed, the dividend and retained earnings coefficients of Model II should be the same. However, with the exception of chemicals-1951 the difference between the coefficients is statistically significant. Durand's bank stock study presents the same picture on this question.[11]

Since the proposition that the rate of profit at which a common stock sells is independent of the dividend rate has some intuitive merit, a theoretical explanation of the statistical findings presented above is of interest. The first point to be noted is that the dividend hypothesis is correct regardless of whether the earnings hypothesis is correct. The only point at issue is whether the dividend hypothesis is unnecessary. Can one study the pricing of common stocks and related questions without considering the fraction of income paid in dividends? It is therefore possible to investigate the problem by using a more rigorous formulation of the dividend hypothesis to establish the condition for the validity of the earnings hypothesis.

Let k be the rate of profit at which a stock is selling, Y_t the income expected in year t, b the fraction of income the corporation is expected to retain, and r the rate of profit it is expected to earn on investment. The corporation's dividend is expected to grow at the rate br, and the price of the stock at $t = 0$ is:

$$P_0 = \int_0^\infty (1-b) Y_t e^{-kt}\, dt = \int_0^\infty (1-b) Y_0 e^{brt} e^{-kt}\, dt. \tag{4}$$

The price of the share is finite and the integration may be carried out if $k > br$, in which case

$$P_0 = \frac{(1-b)}{k-br} Y_0. \tag{5}$$

It may be noted that if $k = r$, equation (5) reduces to

$$P_0 = \frac{1}{k} Y_0, \tag{6}$$

but this is not relevant to the question at issue. For the earnings hypothesis to be valid, it is necessary that k be independent of b. That is, the rate of profit required by the market should be independent of the fraction of income retained.

We could reason as follows. A necessary condition for the price of a stock to be finite is $k > br$. This condition is most easily satisfied if k is an increasing function of br, and if this is true we would also expect that k will vary with b. Other things equal, the rate of profit required on a common stock will vary for a corporation and among corporations inversely with the dividend rate.

An argument with considerably more theoretical content can be derived from the two following assumptions, both of which appear reasonable. (1) The rate at which a future payment is discounted increases with its uncertainty; and (2) the uncertainty of a future payment increases with the time in the future at which it will be received. It follows that *the* rate of profit at which a stream of expected payments is discounted is really an average of rates, each weighted by the size of the payment. The larger the distant payments relative to the near payments, the higher the average rate that equates the stream of payments with the price, the latter obtained by discounting each future payment at its appropriate rate. The relative size of the distant payments will of course vary with the rate of growth. Therefore, given the current earnings, the rate of profit required on a share increases with the fraction of income retained. The same reasoning provides an explanation for the tendency of interest rates on bonds to in-

[10] This appears to be a widely held point of view in the economics literature. See for example Lutz and Lutz, *The Theory of Investment of the Firm* (Princeton, 1951). The question is nowhere considered explicitly, but it is implicit in the material treated on pages 155 ff.

[11] Durand, op. cit., 10–11.

crease, other things being the same, with the maturity of the bond.

Refinements in the Model

Equation (3) is an extremely simple and crude expression of the dividend hypothesis, and insofar as the values of the coefficients are suspect, it may be due to limitations of the model. In this section we shall discuss the more important limitations, suggest how they may be dealt with, and then present data for a model that attempts to overcome some of these limitations.

1. Correlation between the variables and variation in the coefficients among industries is due in part to the scale factor. The problem may be stated as follows. Assume a sample of n corporations for all of which the dividend is the same, the price differs among the shares, and the average of the prices is higher than the dividend. There is no correlation between dividend and price. However, if n numbers are selected at random and the price and dividend of each share is multiplied by one of these numbers, correlation between the variables will be created. Further, if each of the n random numbers is first multiplied by a constant greater than one, the correlation and the regression coefficient will be larger the larger the value of this constant. The presence of so-called high-priced and low-priced stocks in a sample reflects in some part this scale factor. It is possible that by deflating the data, say by book value, and/or using logs we will moderate the influence of scale on the coefficients.

2. The independent variables in equation (3) are the current values of dividends and retained earnings. These quantities are of interest, however, only because they represent the latest available information for the prediction of future dividends. Insofar as these current values depart from averages over some prior period for extraordinary reasons, investment analysts maintain that the changes should be discounted to arrive at what might be considered normal values. This suggests that some combination of current values and averages over a prior period for dividends and retained earnings would provide a superior explanation of the variation in price among shares.

3. The value the market places on a dividend expectation derived from past dividends and retained earnings may be expected to vary among corporations with the confidence in the dividend stream. This would suggest that the price of a share varies with other variables such as the size of the corporation, the relation of debt to equity, and the stability of its earning record. Insofar as the values of these variables vary among industries, failure to include them introduces variation and error in the dividend and retained earnings coefficients.

4. In the present model the variation in price with growth in the dividend is estimated by using an index of growth, retained earnings, as the independent variable. A model in which it is possible to use the rate of growth itself might yield better results. More important, the definition of the rate of growth has considerable theoretical merit — to date nothing superior has been proposed — but there are empirical problems involved in using it. Variation in accounting practice among firms makes the use of book value as a measure of return on investment questionable. Also, the instability of corporate retained earnings and the possibility that they vary over time differently among industries may make the use of past values to predict the future an heroic assumption. This is particularly true if investors give considerable weight, rationally or otherwise, to other variables in predicting future earnings.

Table 3 presents the regression statistics for the following model

$$P = \beta_0 + \beta_1 \bar{d} + \beta_2 (d-\bar{d}) + \beta_3 \bar{g} + \beta_4 (g-\bar{g}). \qquad (7)$$

In this equation:

P = year-end price divided by book value,
$\bar{d}$ = average dividend for the prior five years divided by book value,
d = current year's dividend divided by book value,
$\bar{g}$ = average retained earnings for the prior five years divided by book value,
g = current year's retained earnings divided by book value.

The deflation by book value was undertaken to eliminate the scale effect discussed previously.[12] The objective was only partially accomplished, since correlation exists between the

[12] The use of deflated variables in regression analysis is a debatable question. See David Durand, op. cit., 56; and

TABLE 3. — REGRESSION OF PRICE ON DIVIDEND, RETAINED EARNINGS, CHANGE IN DIVIDEND, CHANGE IN RETAINED EARNINGS, ALL DEFLATED BY BOOK VALUE

Sample	Constant term	Coefficient and standard error of $\bar{d}$	$d-\bar{d}$	$\bar{g}$	$g-\bar{g}$	Multiple correlation
1951 – Chemicals	−.23	12.42	9.79	18.74	14.36	.80
		(2.63)	(5.98)	(5.96)	(5.60)	
Foods	.04	14.04	8.06	3.16	4.57	.90
		(1.04)	(2.49)	(1.39)	(1.58)	
Steels	.15	9.88	6.38	1.45	.41	.88
		(1.05)	(1.87)	(1.09)	(1.06)	
Machine tools	.12	12.62	5.93	.12	1.11	.91
		(1.17)	(2.75)	(.99)	(.80)	
1954 – Chemicals	.54	17.38	12.71	.12	3.44	.79
		(2.92)	(8.93)	(6.39)	(4.78)	
Foods	−.03	15.51	8.74	5.15	5.96	.92
		(1.04)	(2.82)	(1.66)	(1.67)	
Steels	.18	9.69	3.85	2.02	2.85	.91
		(.99)	(1.13)	(.68)	(.67)	
Machine tools	.05	11.65	6.06	3.70	1.92	.87
		(1.16)	(1.74)	(1.12)	(1.04)	

deflated and undeflated variables. For instance, correlation between P and p for the eight samples ranged from zero to .65 and was more than .4 for six of the samples.

The use of $\bar{d}$ and $(d-\bar{d})$ assumes that the investor values a stock on the basis of the average dividend during the prior five years and the amount by which the current value differs from this average. The same reasoning applies to $\bar{g}$ and $(g-\bar{g})$, which by the way should be interpreted as deflated retained earnings and not as growth rates in the context of this model. The coefficients β_i may be interpreted as follows: $\beta_1 = \beta_2$ (or $\beta_3 = \beta_4$) implies that the investors ignore the average dividend for the prior five years and consider only the current dividend; $\beta_2 = 0$ implies that the current dividend is ignored; $\beta_1 > \beta_2$ implies that investors adjust to a change in the dividend with a lag,[13] i.e., the elasticity of expectations is less than one. The opposite is true if $\beta_1 < \beta_2$.

Turning to the data of Table 3 we see that five of the eight multiple correlation coefficients are lower than in Table 2, and for some the difference is large. This is due to the deflation by book value. For dividends, deflation and/or the use of both the average value and the departure from average appears to have done some good. The range of the dividend coefficient has been reduced by comparison with Table 2, and the change in dividend coefficient is interesting. All but the chemicals coefficients are significant at the five per cent level, and they all are less than the $\bar{d}$ coefficients. Therefore, as expected, a rise in the dividend is discounted until the average has risen to the new level.

The growth coefficients, however, are disappointing. First, the values for $\bar{g}$ are if anything poorer than the values for $Y-D$ in Table 2. Second, three of the eight coefficients are not statistically significant at the five per cent level. Third, for some of the samples $\beta_4 \geqq \beta_3$, which means that investors are either indifferent to past performance or prefer a share for which retained earnings has increased to one for which it has fallen.

The performance of the model just discussed in explaining the variation in price among stocks is far superior to the simple empirical approach presented earlier. However, considerable room for improvement remains. The lines along which it will be realized appear to be a more effective representation of growth and the recognition of variables which influence the valuation of a dividend expectation. Solution of the scale problem through a different structural relation among the variables may also be of value.

Edwin Kuh and John R. Meyer, "Correlation and Regression Estimates when the Data are Ratios," *Econometrica*, XXIII (October 1955), 400–16.

[13] We are talking about an unexpected change in the dividend, since d is the percentage that the dividend bears to book value. A rise in the dividend proportional to the rise in book value counts as no change in the dividend.

Optimal Investment and Financing Policy
Author(s): M. J. Gordon
Reviewed work(s):
Source: *The Journal of Finance*, Vol. 18, No. 2 (May, 1963), pp. 264-272
Published by: Wiley-Blackwell for the American Finance Association
Stable URL: http://www.jstor.org/stable/2977907

MANAGEMENT OF CORPORATE CAPITAL

OPTIMAL INVESTMENT AND FINANCING POLICY*

M. J. GORDON†

IN TWO PAPERS[1] AND IN a recent book[2] I have presented theory and evidence which lead to the conclusion that a corporation's share price (or its cost of capital) is not independent of the dividend rate. As you may know, MM (Modigliani and Miller) have the opposite view, and they argued their position at some length in a recent paper.[3] Moreover, the tone of their paper made it clear that they saw no reasonable basis on which their conclusion could be questioned. Since they were so sure of their conclusion, it would seem advisable for me to review carefully my thinking on the subject, and this meeting appears to be a good time and place to do so.

I

Let us begin by examining MM's fundamental proof that the price of a share is independent of its dividend. They defined the value of a share at $t = 0$ as the present value of (1) the dividend it will pay at the end of the first period, D_1, plus (2) the ex-dividend price of the share at the end of the period, P_1:

$$P_0 = \frac{1}{1+k}[D_1 + P_1]. \tag{1}$$

They then asked what would happen if the corporation, say, raised its dividend but kept its investment for the period constant by selling the additional number of shares needed to offset the funds

* This paper and the following papers by Ezra Solomon, James E. Walter, and John Lintner, with discussions by Herbert Dougall, Merton Miller, and Robert F. Vandell, were presented at a meeting of the American Finance Association in Pittsburgh, Pa., on December 29, 1962. The program was under the chairmanship of J. Fred Weston.

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1. "Dividends, Earnings and Stock Prices," *Review of Economics and Statistics,* May, 1959, pp. 99–105; "The Savings, Investment and Valuation of the Corporation," *ibid.,* February, 1962, pp. 37–51.

2. *The Investment Financing and Valuation of the Corporation* (Homewood, Ill.: R. D. Irwin, 1962).

3. "Dividend Policy, Growth, and the Valuation of Shares," *Journal of Business,* October, 1961, pp. 411–33.

lost by the dividend increase. They demonstrated that the ex-dividend price of the stock at the end of the period would go down by exactly the same amount as the increase in the dividend. Since the sum $D_1 + P_1$ remains the same, P_0 is unchanged by the change in the dividend.

I will not review their proof of the theorem in detail because I find nothing wrong with it under the assumption they made that the future is certain. However, after proving the theorem a number of times under different conditions, they withdrew the assumption of certainty and made the dramatic announcement, "our first step, alas, must be to jettison the fundamental valuation equation."[4] Under uncertainty, they continued, it is not "at all clear what meaning can be attached to the discount factor. . . ."[5] The implication which they made explicit in discussing my work is that under uncertainty we cannot represent investors as using discount rates to arrive at the present value of an expectation of future receipts.

It would seem that all is lost. But no! On the very next page we are told that their "fundamental conclusion need not be modified merely because of the presence of uncertainty about the future course of profits, investment, or dividends. . . ."[6] By virtue of the postulates of "imputed rationality" and "symmetric market rationality," it remains true that "dividend policy is irrelevant for the determination of market prices."[7]

Their paper continued with a discussion of market imperfections, in which they note that the most important one, the capital gains tax, should create a preference for low payout rates. They concede that it may nevertheless be true that high payout rates sell at a premium, but they found ". . . only one way to account for it, namely as a result of systematic irrationality on the part of the investing public." They concluded with the hope that ". . . investors, however naive they may be when they enter the market, do sometimes learn from experience; and perhaps, occasionally even from reading articles such as this."[8]

It would seem that under uncertainty they might have been less sure of their conclusion for two reasons. First, under uncertainty an invester need not be indifferent as to the distribution of the one-period gain on a share between the dividend and price appreciation. Since price appreciation is highly uncertain, an investor may prefer

4. Miller and Modigliani, *op. cit.*, p. 426.

5. *Ibid.*, p. 427.

6. *Ibid.*, p. 428.

7. *Ibid.*, p. 429.

8. *Ibid.*, p. 432.

the expectation of a \$5 dividend and a \$50 price to a zero dividend and a \$55 price without being irrational. Second, the expectation of a stock issue at $t = 1$ may have a depressing influence on the price at $t = 0$. What MM did was both change the dividend and change the number of new shares issued. Can we be so sure that the price of a share will not change when these two events take place?

II

Let us turn now to the proof of the MM position on the dividend rate that I presented in my *RES* paper and book. The reasons for presenting this proof will be evident shortly. Consider a corporation that earned Y_0 in the period ending at $t = 0$ and paid it all out in dividends. Further, assume that the corporation is expected to continue paying all earnings in dividends and to engage in no outside financing. Under these assumptions the company is expected to earn and pay Y_0 in every future period. If the rate of return on investment that investors require on the share is k, we may represent the valuation of the share as follows:

$$P_0 = \frac{Y_0}{(1+k)^1} + \frac{Y_0}{(1+k)^2} + \frac{Y_0}{(1+k)^3} + \ldots + \frac{Y_0}{(1+k)^t} + \ldots. \quad (2)$$

We may also say that k is the discount rate that equates the dividend expectation of Y_0 in perpetuity with the Price P_0.

Next, let the corporation announce at $t = 0$ that it will retain and invest $Y_1 = Y_0$ during $t = 1$ and that it expects to earn a rate of return of $k = Y_0/P_0$ on the investment. In each subsequent period it will pay all earnings out in dividends. Share price is now given by the expression

$$P_0 = \frac{0}{(1+k)^1} + \frac{Y_0 + kY_0}{(1+k)^2} + \frac{Y_0 + kY_0}{(1+k)^3} + \ldots + \frac{Y_0 + kY_0}{(1+k)^t}. \quad (3)$$

Notice that the numerator of the first term on the right side is zero. It is the dividend and not the earnings in the period, since the investor is correctly represented as using the dividend expectation in arriving at P_0. If he were represented as looking at the earnings expectation, then as Bodenhorn[9] noted, he would be double-counting the first period's earnings.

It is evident that, as a result of the corporation's decision, the investor gives up Y_0 at the end of $t = 1$ and receives, in its place, kY_0

9. Diran Bodenhorn, "On the Problem of Capital Budgeting," *Journal of Finance*, December, 1959, pp. 473–92.

in perpetuity. The distribution of dividends over time has been changed. It is also evident that kY_0 in perpetuity discounted at k is exactly equal to Y_0. Hence P_0 is unchanged, and the change in the distribution over time of the dividends had no influence on share price. In general, the corporation can be expected to retain and invest any fraction of the income in any period without share price being changed as a consequence, so long as r, the return on investment, is equal to k. If $r > k$ for any investment, P_0 will be increased, but the reason is the profitability of investment and not the change in the time distribution of dividends.

Assume now that when the corporation makes the announcement which changes the dividend expectation from the one given by equation (2) to the one given by equation (3), investors raise the discount rate from k to k'. For the moment let us not wonder why the discount rate is raised from k to k', i.e., why the rate of return investors require on the share is raised as a consequence of the above change in the dividend expectation. If this takes place, equation (3) becomes

$$P_0' = \frac{0}{(1+k')^1} + \frac{Y_0 + kY_0}{(1+k')^2} + \frac{Y_0 + kY_0}{(1+k')^3} + \ldots + \frac{Y_0 + kY_0}{(1+k')^4} + \ldots . \quad (3a)$$

It is clear that with $k' > k$, $P_0' < P_0$.

Let us review what happened. The dividend policy changed: the near dividend was reduced, and the distant dividends were raised. This caused a rise in the discount rate, and the result was a fall in the price of the share. I, therefore, say that the change in dividend policy changed the share's price.

In response to this argument, MM stated that I fell into "the typical confounding of dividend policy with investment policy."[10] I don't understand their reasoning. It is well known that when the rate of return on investment is set equal to the discount rate, changing the level of investment has no influence on share price. By this means, I neutralized the profitability of investment. It seems to me perfectly clear that I did not confound investment and dividend policy; I changed the discount rate. Share price changed with the dividend rate in the above example because the discount rate was changed. The issue, therefore, is whether the behavior of investors under uncertainty is correctly represented by a model in which the discount rate that equates a dividend expectation with its price is a function of the dividend rate.

10. Miller and Modigliani, *op. cit.*, p. 425.

I cannot categorically state that k is a function of the rate of growth in the dividend, i.e., the dividend rate, but I can present some theoretical considerations and empirical evidence in support of the theorem. It seems plausible that (1) investors have an aversion to risk or uncertainty, and (2), given the riskiness of a corporation, the uncertainty of a dividend it is expected to pay increases with the time in the future of the dividend. It follows from these two propositions that an investor may be represented as discounting the dividend expected in period t at a rate of k_t, with k_t not independent of t. Furthermore, if aversion to risk is large enough and/or risk increases rapidly enough with time, k_t increases with t.

It is therefore possible, though not certain, that investor behavior is correctly approximated by the statement that, in arriving at the value of a dividend expectation, they discount it at the rates k_t, $t = 1, 2 \ldots$, with $k_t > k_{t-1}$. In this event the single discount rate we use in stock value models is an increasing function of the rate of growth in the dividend. In short, dividend policy influences share price. To illustrate the conclusion, let us rewrite equation (2):

$$P_0 = \frac{Y_0}{(1+k_1)^1} + \frac{Y_0}{(1+k_2)^2} + \frac{Y_0}{(1+k_3)^3} + \ldots \frac{Y_0}{(1+k_t)^t} + \ldots . \quad (4)$$

We now look on the k of equation (2) as an average of the k_t of equation (4) such that if the entire dividend expectation is discounted at this single rate, it results in the same share price. The discount rate k is an average of the k_t with Y_0, the weight assigned to each item.

Once again let the corporation retain $Y_1 = Y_0$ and invest it to earn kY_0 per period in perpetuity. Using the sequence of discount rates k_t, the same as that appearing in equation (4), the valuation of the new dividend expectation becomes

$$P_0' = \frac{0}{(1+k_1)^1} + \frac{Y_0 + kY_0}{(1+k_2)^2} + \frac{Y_0 + kY_0}{(1+k_3)^3} + \ldots + \frac{Y_0 = kY_0}{(1+k_t)^t} + \ldots . \quad (5)$$

The shareholder gives up Y_0 and gets kY_0 in perpetuity, but the latter is now discounted at the rates k_t, $t = 2 \rightarrow \infty$, and it can be shown that kY_0 so discounted is less than Y_0. Hence $P_0' < P_0$, and dividend policy influences share price. It also can be shown that k', the new average of the same k_t, is greater than k. In general, reducing the near dividends and raising the distant dividends (lowering the dividend rate) changes the weights of the k_t and raises their average.

III

To summarize the theoretical part of my argument, I started with two assumptions: (1) aversion to risk and (2) increase in the uncertainty of a receipt with its time in the future. From these assumptions I proceeded by deductive argument to the proposition that the single discount rate an investor is represented as using to value a share's dividend expectation is an increasing function of the rate of growth in the dividend. The consequence of the theorem is that dividend policy per se influences the value of a share. The assumptions have enough intuitive merit, I believe, that the theorem may in fact be true.

Before proceeding to the empirical evidence, I would like to comment briefly on two other criticisms MM directed at my argument. First, they differentiated between my "purely subjective discount rate and the objective market-given yields" and stated: "To attempt to derive valuation formulas from these purely subjective discount factors involves, of course, an error. . . .[11] My assumptions and empirical results may be questioned, but where is the error? Does the theorem fail to follow from the assumptions? Why, as they suggest, is it logically impossible for an investor to arrive at the value of a share by estimating its future dividends and discounting the series at a rate appropriate to its uncertainty?

The following MM criticism of my argument I find even more confusing. They stated: "Indeed if they [investors] valued shares according to the Gordon approach and thus paid a premium for higher payout ratios, then holders of the low payout shares would actually realize consistently higher returns on their investment over any stated interval of time."[12] Under this reasoning two shares cannot sell at different yields regardless of how much they differ in risk because the holders of the higher-yield share would "actually realize consistently higher returns over any stated interval of time." Do MM deny that investors have an aversion to risk?

To test the theorem empirically, I proceeded as follows. The valuation of a share may be represented by the expression

$$P_0 = \int_0^\infty D_t e^{-kt} dt, \tag{6}$$

where D_t is the dividend expected in period t and k is an operator on the D_t that reduces them to their present value to the investor.

11. *Ibid.*, p. 424. 12. *Ibid.*, p. 425.

Equation (6) is a perfectly general statement that is not open to question. However, to use the equation in empirical work, we must specify how investors arrive at D_t from observable variables. For this, I assumed that investors expect a corporation will: (1) retain the fraction b of its income in each future period; (2) earn a rate of return, r, on the common equity investment in each future period; (3) maintain the existing debt-equity ratio; and (4) undertake no new outside equity financing. Under the above assumpions the current dividend is $D_0 = (1 - b)Y_0$, and its rate of growth is br. Further, the entire dividend expectation is represented by these two variables, and equation (6) is equal to

$$P_0 = \frac{(1-b)Y_0}{k - br}. \tag{7}$$

The above four assumptions may be criticized as being too great a simplification of reality. I have admitted their limitations, and I welcome improvement, but I know of no other empirical model that contains as rich and accurate a statement of the dividend expectation provided by a share. Most empirical work, including the published work of MM, represents the investor as expecting that the corporation will pay all earnings in dividends and engage in no outside financing. They, therefore, also ignore the influence of the profitability of investment on share price. This model incorporates a prediction of the corporation's investment and rate of return on the investment in each future period. The expected investment in period t is the fraction b of the period's income plus the leverage on the retention that maintains the corporation's existing debt-equity ratio. Further, the influence of this retention and borrowing on the dividend expectation is incorporated in the model.

The interesting thing about the model as it stands is that it is consistent with the MM position and should provoke no objection. To see this, let us make their assumption that k is independent of b and, to neutralize the profitability of investment, let $r = k$. In this model, dividend policy is represented by b the retention rate, so that, if we take the derivative of P_0 with respect to b, we establish the relation between share price and the dividend rate. We find that $\delta P/\delta b = 0$. The value of a share is independent of the dividend rate—exactly what MM argue.

One can use this model in empirical work under the assumption that k is independent of br. I did and obtained poor results. Since I found good theoretical grounds for believing that k is an increasing function of *br,* it would seem reasonable to explore the hypothesis, and that is what I did. If k is an increasing function of *br,* we can write equation (7) as

$$P_0 = A_0[(1-b)Y_0][1+br]^{\alpha_2}. \tag{8}$$

In this expression, A_0 represents the influence of all variables other than the current dividend, $(1 - b)Y_0$, and its rate of growth, *br*. When $b = \delta$, P_0 is the multiple A_0 of Y_0. As *br* increases, the dividend, $(1 - b)Y_0$, falls and *br* rises, the former lowering price and the latter raising price. Whether P_0 rises or falls with b depends on r, the profitability of investment, and on α_2. The expression α_2 may be looked on as how much investors are willing to pay for growth. Its value depends on how fast the k_t rise with t, that is, on how fast uncertainty increases with time and on the degree of investor aversion to risk.

It should be noted that equation (8) is not merely a stock value model. Given the investor's valuation of a share, A_0 and α_2, and, given the profitability of investment, r, the model may be used to find the retention rate (equal to the investment rate under our assumptions) that maximizes the value of a share. Extensions of the model developed elsewhere[13] allow its use to find the investment and the financing, retention, debt, and new equity that maximize share price.

The empirical results I obtained with the above model have been published in detail,[14] and all I will say here is that they are very good. Although the results compare favorably with earlier work, they are not good enough to settle the question. MM[15] and Benishay[16] have pointed out that my independent variables are not free of error, and the consequence is that the parameter estimates have a downward basis. Kolin[17] has reported that his empirical work revealed no relation between dividend policy and share price. As things stand, I would say that the influence of dividend policy on share price is a question that requires further study. The axiomatic basis

13. M. J. Gordon, *The Investment, Financing and Valuation of the Corporation* (Homewood, Ill.: R. D. Irwin, 1962).

14. *Ibid.*

15. Franco Modigliani and Merton Miller, "The Cost of Capital Corporation Finance, and Theory of Investment: Reply," *American Economic Review,* September, 1959, pp. 655–69.

16. Haskel Benishay, "Variability in Earnings-Price Ratios: Reply," *American Economic Review,* March, 1962 pp. 209–16.

17. Marshal Kolin, *The Relative Price of Corporate Equity* (Boston: Harvard Business School).

of the MM position is certainly not so powerful as to force the acceptance of their conclusions.

IV

I should like to close with a brief comment on the two major camps that are emerging with respect to the theory of corporation finance. In both camps optimal policy is taken as the policy that maximizes the value of the corporation. Although corporations may not make investment and financing decisions with only this objective in mind, managements are certainly not indifferent to the prices at which their corporations' securities sell. Hence the policy question posed has practical significance.

In one camp, where we find MM, it is argued that a corporation's cost of capital is a constant—i.e., independent of the method and level of financing. Optimal policy is the investment that equates the marginal return on investment with this cost of capital. The inescapable conclusion is that financing policy is not a problem. The opposite position is that a corporation's cost of capital varies with the method and level of financing. My judgment is that the theoretical and empirical evidence we have favors this position.

However, regardless of which view prevails, the battle should be lively and productive. For a long time the position that cost of capital is a constant was held almost exclusively by economists, who were sophisticated in methods of theoretical and econometric analysis but knew little of finance. By contrast, the position that the cost of capital is a variable was held by finance men, who were familiar with their subject but not with advanced methods of theoretical and empirical research. People in each group talked only to those who agreed with them, and in consequence not much was said. The situation has changed, it will change further, and the promise is that the lively debate and active research in progress will advance our knowledge on the subject.

THE COST OF CAPITAL AND OPTIMAL FINANCING OF CORPORATE GROWTH

JOHN LINTNER*

THE INTEREST OF PROFESSIONAL economists in the theory of corporate finance and capital budgeting has increased markedly within the last decade.[1] Nevertheless, the literature is still marked by confusion and even contradiction: the decision rules which have been proposed for determining the optimal capital budget in a corporation and its optimal capital structure and reliance on different sources of financing are mutually inconsistent in the sense that they would lead to (often substantially) different decisions under given sets of circumstances.

None of the marked differences in decision rules advanced in the literature reviewed here can be attributed to different assumed *goals,* since all the authors to be cited have, explicitly or implicitly, offered their respective criteria as the means to accomplish the same ultimate objective—the greatest satisfaction of common stockholders' preferences. Moreover, since increased current share valuations *ceteris paribus* obviously increase shareholders' current wealth, which in turn clearly implies greater utility, this criterion of optimizing shareholders' utility has in practice been identified with the maximization of the current market value of the common stock. Further, all authors assumed maximizing behavior to be universal and financial markets to be purely competitive. *These premises and specifications are accepted without question and maintained throughout the present paper.*

I. INTRODUCTION

Disagreements on optimal size of capital budgets and cost of capital.—The seriousness of the conflicts in the literature on the theory of corporate finance and capital budgeting is clearly indicated in

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1. This paper is one part of a series of interrelated theoretical and statistical studies of corporate and financial policies being made at the Harvard Business School under a grant of the Rockefeller Foundation for work in the general area of profits in the functioning of the economy. The Foundation's generous support for this work is most gratefully acknowledged. Major parts of this paper are based upon the longer manuscripts [*a*18], [*b*8], [*b*9], and [*b*10]. (The coverage of [*a*19], as previously announced, was cut back to [*b*8].)

the markedly different conclusions offered by eminent economists regarding the determination of the optimal size of capital budgets.

The Lutzes, in their classic study a decade ago [*a*20],[2] concluded that investment within the firm should be increased up to the point where this course no longer added more to the collective stockholders' "net profits prospects" than further outside investment. Cash flows from borrowing and debt service are to be deducted from those of the internal investment plan, the resulting stream of net cash flows is to be discounted at the yield of the preferred maturity of outside riskless investment (government bonds), and internal investments are to be increased only so long as the certainty equivalents of the resulting present values exceed the cost of the investment.

Roberts [*a*28] concurs in the use of the outside lending rate and the netting of cash flows from borrowings and repayments, but argues that the discount rate should be the external yield available on outside investments having (subjectively) similar risk, and he equates this with the *current earnings yield* of the company's own stock.[3] His decision rule is: Investments are to be made so long as the present value of prospective incremental receipts exceeds that of incremental cash outflows, when both flows are discounted at a rate equal to the current earnings yield on the stock. The relevant investment fund flows are the same for the Lutzes and Roberts in any given case; but Roberts' discount rate is much greater, and it has not been shown that this difference offsets the Lutzes' utility adjustment of present values to certainty equivalents.

Dean [*a*4] and, more recently, Modigliani-Miller [*a*23; *a*25], Kuh [*a*16], Benishay [*b*3], and Weston [*a*34; *b*15] have also capitalized corporate earnings to determine market values and have all argued that the current earnings yield on common stock is the proper discount rate when no debt is outstanding, but otherwise they urge the use of a *current-market-value weighted average cost of debt and equity capital* as the proper discount rate. This is often a substantially *lower* figure than the current earnings yield on the equity when debt is outstanding;[4] and these authors do not net debt charges from investment fund flows. For given investment projects, the relevant fund flows for these authors are larger than for Lutz and Roberts;

2. Especially chaps. xiii–xvii.

3. Spencer and Siegelman [*b*13] have recently advocated the same rule with the proviso that the earnings yield should be measured as it would be "when the firm has what the market considers to be a well-balanced capital structure."

4. This is true even when the market-value weights urged by Modigliani and Miller are used; further differences are produced by Dean's advocacy of book-value weights; but, for reasons already clear in the literature, this latter position is invalid.

their further use of a lower discount rate when debt is outstanding clearly implies acceptance of projects (and thereby extensions of the size of the capital budget) which would not be made under Roberts' rule (and presumably under the Lutzes).

Similarly, Solomon [*a*30], like Roberts, advises netting cash flows due to borrowing from those of individual investment projects, but he substitutes the ratio of "estimated future average earnings per share" to current market price as his recommended discount rate. In growing companies this is an even *higher* figure than the current earnings yield on equity. For Solomon and Roberts, the relevant investment fund flows from given projects are the same (i.e., both deduct interest costs when debt is used in financing the project), but Solomon's rule will reject projects that Roberts' rule would accept in growth situations because of the latter's lower discount rate.

A still different rule has been advanced by Walter [*a*33] who advocated discounting investment opportunities at the rate at which current and future dividends are capitalized; this rate being defined as "the underlying yield on safe securities (government bonds?) and the required risk premiums."[5] Similarly, Bodenhorn [*a*2] has also urged the use of the market discount rate for comparable risk, and Modigliani and Miller in a new paper [*a*25] have also fixed upon the market discount rate as the proper cost of capital.[6] In some contexts (see below, *passim*), growth opportunities will make current earnings yields less than current market discount rates, and these authors' rule would lead to rejection of projects which Roberts' rule would accept, and they would correspondingly reduce the size of capital budgets and the rate of growth below the levels his rule would justify. In other contexts, the opposite would occur. The rule advanced by Shapiro and Gordon [*a*14], based upon the sum of the current dividend yield and the expected growth rate, would in general lead to still different decisions, and, as our final illustration, we note that Gordon in later writings [*a*11–13; *b*4] has advanced a still different requirement.

5. Since Walter ignores borrowing, strict comparison of his rule with that of other authors can be made only in situations where there is no borrowing; but the conflict in the decisions implied on given sets of data is clear in this class of cases. If different decisions will be made in non-leverage cases, the rules necessarily have different implications in general.

6. They have thus abandoned the identification of the market discount rate with current earnings yield *in the absence of debt* which provided the decision rule in their earlier paper [*a*23]. In the presence of growth opportunities, they agree with Solomon that the relevant cost-of-capital is greater than current earnings yields, but their figure is lower (and very much lower in strong growth situations) than his ratio of *future average* earnings to current prices and will thus accept many projects he would reject.

So far we have emphasized differences in rules for accepting investments and setting the optimal size of the capital budget. We should also note that the various authors differ on whether—and how—their respective preferred "cost-of-capital" figure varies as a function of the existing capital structure (primarily the mix of debt and equity capital) and also as a function of the form of the new financing to be used for the capital budget—the proportion of retained earnings, additional debt, and/or new issues of equity capital.

Modigliani-Miller [*a*23 and *a*25] take the limiting position that (apart from the relatively small discrimination in favor of debt financing under the corporate income tax) the cost of capital is *independent* of both the existing capital structure and the mix of new financing, a position apparently also shared by Dean. Others—notably Solomon, Kuh, Weston, Gordon, Duesenberry [*a*6], Schwartz [*b*10], and myself [*a*17, 18 and *b*7]—argue that the cost of capital *is* a function of the financing mix, although, once again, there are substantial differences in the exact form of the dependence. Indeed, the rules for decisions regarding *how* the investments should be financed differ as seriously as those for determining the size of the capital budget itself—i.e., those determining the *amount* of finance (whatever the type) to be used.

Since all these authors have defined their optima in terms of maximization of the current market value of existing equity issues, all these differences in the decision rules come down fundamentally either to differences in assumptions regarding the character of the corporations investment opportunities themselves (to which we revert below) or to differences in the models the various authors have used to explain (*a*) the determination of stock-market prices when there is no debt outstanding and (*b*) the effects of leverage on those prices. Indeed, in the latter two respects, the more significant differences can be traced to the respective author's choice of one of two basic assumptions within each of the two categories just noted: specifically, to whether or not (1) [as alleged in "pure earnings" theories], *ceteris paribus*, the valuation of unlevered equities is determined by (expectational) current earnings *independent* of dividends and (2) [as held in "entity value" theories] the market valuation of the corporate entity is independent of its capitalization, apart from corporate tax differentials due to the deductibility of debt interest.

Further context of present paper.—In the usual "theory of the firm," there are two separate (or at least separable) parts to the analysis: (*a*) given production functions and supply conditions in factor markets, how can the firm minimize the cost of producing each possible quantity of output? and (*b*) given the results of such isoquant-cum-budget-line analysis and specified product market conditions, what *quantity* of output produced and sold will maximize profit? The necessary and sufficient conditions for the validity of the "pure earnings and company investment" and "entity value" theories can best be analyzed under an assumption that time vectors of investment budgets and corporate earnings are fixed throughout all time independent of dividends and the finance mix.

These issues were examined in detail in a previous paper [*b*7]. That analysis corresponded in our financial context to the "theory of the production of a given output" ("output" here being vectors of capital budgets and their associated earnings). Corresponding to the second major problem in the standard theory of the firm, there is the further major issue for the theory of corporate financing and capital budgeting: "given the minimum 'cost' (i.e., optimal finance mix) for each possible size of capital budget, what is the optimal size of the capital budget under any given functional relationship between size-of-budget and corporate earnings?" This latter issue is the primary focus of the present paper.

Some of the central results of the previous paper, however, obviously provide an essential basis for the present one: specifically, that both the "pure earnings" theory (investors indifferent to particular dividend vectors) and the "entity value" theory (the sum of market values of equity and debt invariant to debt) *are invalid* even with the time vectors of earnings and investments fixed forevermore *if* the market context involves (1) costs of issuing securities, *or* (2) any personal tax differentials, *or* (3) any lack of prescience *and* identity in investors' subjective probability distributions, *or* (4) any combination of them;[7] that the model making stock prices depend *essentially* upon the (present values of the) time vectors of *cash dividend flows to investors* remains valid even under these fully generalized neoclassical conditions, while the alternatives are valid *if and only if* stated in forms identically reducible to this dividend theory; and that the significance of time vectors of earnings (and of company investments) lies in its implications *for* the prospective stream of dividends, rather than vice-versa.

In the present paper, I consequently rely essentially on "present

7. Also, of course, any corporate tax differentials between interest and other income will invalidate the "entity value" theory.

values of dividends models"[8] and, removing the constraint of fixed investment budgets, examine optimal decision rules for the finance mix and size of the capital budget of a corporation and its optimal (expected average) rate of growth over time. Since all these matters depend on the proper determination of the relevant "cost of capital," this issue also provides a common concern throughout the paper.

In keeping with space limitations and the interests of this group, this paper will focus on the important, but necessarily limited, objective of setting forth the essential logic of some of the more fundamental conclusions I have reached on these issues. To this end, I will outline the basic structure of some of the more useful analytical models I have been developing, present some rigorous proofs, and motivate others. A full set of rigorous mathematical derivations and proofs and a more complete and general analysis of these and related issues will be found elsewhere. Among other simplifications, I shall assume throughout this paper that all tax rates are zero; that the (riskless) discount rates $k_T = k$ are constant over time; and that the variance of profit rates with no growth and no debt $\sigma^2_{p_0}$ is given and constant over time.

Finally, two definitions are needed at the outset which, for convenience, are stated in general form to cover uncertainty—certainty being the limiting case for each when all variances approach zero. Specifically, the *marginal cost of (a given type of) capital* for the corporation is the *minimum* (expectation of) *rate of return required on a marginal investment* for the shareholders to be better off (value of existing equity greater) *with* the incremental-investment-cum-this-incremental-financing *than without either* the increment to the capital budget *or* this financing. Similarly, *corporate earnings* or *profits* (after taxes and interest) for any period are defined to be equal to the maximum cash dividend which (expectationally) *could* be paid in that period consistent with (pro-forma) no outside financing *and* with the *expectation* that a similarly large dividend *could* then be paid in future periods subject to the same constraint (this pro-forma constraint tying the earnings back to earnings on present assets).[9]

II. Unlevered Firms under Certainty

Certain issues can most conveniently be handled under the simplifying assumption of certainty. First of all, it can readily be shown that, *even in the absence of issue costs, taxes, or uncertainty,* the

8. See n. 15, p. 301.

9. See n. 17, p. 250, in [*b7*].

relevant marginal cost of capital for the corporation is not equal to the discount rate k unless (*a*) the "investment opportunity" or "profit" function relating the average rate of internal return,[10] p_t, per dollar of new investment is *strictly independent* of the amounts of investments made in earlier or subsequent periods, *or* (*b*) the profit function at every point in time exhibits strictly constant returns to scale *and* these returns $p_t = k$. The absence of costs, taxes, and uncertainty, together with a profit function $p_t = \psi_t\ (F_t^* \ldots)$ in which ψ_t is strictly independent of the dollar size of the company's aggregate investments (capital budget) F_τ^* for all $\tau \neq t$, are *sufficient conditions* to make the discount rate k the appropriate cost of capital because under the fully idealized neoclassical conditions all marginal rates of substitution for all companies and investors are equal to the discount rate k in equilibrium, as demonstrated by Fisher thirty years ago [*a*10].

Under these very restrictive conditions, all investments are perfect substitutes at the margin, and, in keeping with standard classical theory, the company should include all increments of investment in each period which have a marginal rate of return ρ on their dollar cost $\geq$ the discount rate k. Allowance for issue costs and taxes, however, requires important modifications even under certainty (cf. [*b*7, *a*23, *a*30, and *a*33]), although, with no taxes, the minimum acceptable return $\rho_0 = k$ *so long as* all investments whose $\rho \geq k$ do not exhaust current earnings.[11]

But models based on the profit function ψ_t *restricted* by an independence assumption regarding F_τ^*, $\tau \neq t$, are at best inadequate to handle—and in general[12] are inherently biased with respect to—the essential elements and issues of growth and change over time which constitute the primary focus of this paper. For this restriction on ψ_t implies that the (average *and* marginal) profitability of *any given*

10. In this paper, I shall consistently use decision rules in the form of marginal internal rate of return $\geq$ marginal cost of capital. Under the assumptions made concerning the efficient set of the (portfolio) of investment opportunities facing the firm (and the simplifying assumption of constant discount rates over time), these rules are strictly equivalent to the alternative statement of rules in the form of present values exceeding costs. Cross-sectional non-independence of investment opportunities are subsumed in the efficient opportunity set; perfect capital markets are assumed throughout; and major lumpiness in discrete investment projects causes no trouble when our assumptions regarding the regularity and smoothness of the envelope of the efficient set are satisfied. Cf. [*b*5 and *b*2].

11. The reason is that such costs simply insure that new investments in this range, if made, will be financed by retained earnings. See [*b*7].

12. In this respect, they will be acceptable as a first approximation only for firms selling in markets within which no seller has or creates any significant (product) market power which affects the profitability of future investments, where investments include outlays for product promotion as recognized in Dean [*a*4].

dollar-sized capital budget for, say, IBM, du Pont, Avon, or General Motors in 1962 is independent of the capital investments they have made in the last one, three, five, ten, or even twenty years—which is obviously not true. In particular, this restriction on ψ_t ignores the hard fact that—especially in the major oligopolistic industries which account for such large fractions of plant and equipment expenditures and of total equity values, but also quite generally—the position of a firm in its industry and the profitability of further new investments depend heavily upon whether it has led or lagged in the introduction of new products, new capacity, new cost-reducing technologies, research and development, long-range advertising, and other promotion of product-market position, and so on in the recent and more remote past.[13] All this is true not only in the short run but, cumulatively, in the longer run as well.

To encompass the essence of the problems involved in decisions for continuing growth and to incorporate basic determinants of the profit opportunities available to potentially growing firms at given points in time, the function ψ_t must explicitly depend on investments in other periods (or, as their surrogate, recent realized—or "normalized"—levels of earnings). The central implications of such dependence are brought out most simply in the profit function originally advanced by Preinreich [*a*27] and Williams [*a*35] a quarter of a century ago and more recently also adopted by Gordon-Shapiro and Gordon in which the function ψ_t is invariant over time when written in the form

$$p_t = \psi_t^*(F^*/Y^*, \ldots) = \psi_t^*(f, \ldots) = p(f, \ldots) \\ = p \text{ constant over time,} \tag{1}$$

where Y^* is the corporation's aggregate earnings in the current period, and $f = F^*/Y^*$. Since p will not in general be invariant with respect to f, we also have $p'(f) \leq 0$ but constant over time as a function of f, and there will be a marginal rate of return, defined as

$$\rho = \frac{\delta f p(f)}{\delta f} = p(f) + f p'(f), \tag{1a}$$

13. See Lintner [*a*17], Duesenberry [*a*6], and Meyer-Kuh [*a*22]. The importance of including outlays for advertising, research and development, and other promotion of product-market position in the capital budget, when the outlays are intended to affect receipts in subsequent periods, has been emphasized by Dean.

It should be emphasized that we assume throughout this paper that *financial* markets are strictly and universally *purely competitive* (except for the fact that any given company is the sole issuer of its own securities), but this does not require us to ignore well-known facts of life in the *product*-market place—which do affect in a fundamental way the properties of the firm's profit-opportunity function.

which will also be constant over time for given f, with the further property that $\delta\rho/\delta f \leq 0$.[14]

Since *the criterion* ordering the desirability of alternative outcomes is the market price of the common equity, the profit function (1) must be incorporated into a model of stock price which specifies price as a function of both the profit opportunities of the company and the amounts and types of financing used to finance its internal investments or capital budget. As a first step, note that using continuous compounding for convenience, so that Y_t^* is the instantaneous *rate* of earnings flow, the rate of growth g^* of Y_t^* is

$$g^* = g^*_{Y_t} = d \log Y_t^*/dt = fp(f). \qquad (2)$$

With aggregate dividends determined by $D_t^* = xY_t^*$, where x is the dividend payout ratio, which is a decision variable also assumed to be constant over time, it is clear that the growth rates of dividends and earnings will be equal and that the aggregate dividend distribution at any time t will be

$$D_t^* = xY_t^* = xY_0^* e^{g^*t} .$$

Stock prices at any given time, however, reflect the values of the streams properly attributable to the *then outstanding* shares of stock. Let N_t be the number of shares at time t, and we have $D_t = D_t^*/N_t$ and $Y_t = Y_t^*/N_t$. It follows that if new shares are issued at the relative rate $n = g_{N_t} = d \log N_t/dt$ and we let g without asterisk represent the rate of growth of dividends and earnings on shares outstanding at time t, we have

$$g = d \log D_t/dt = d \log Y_t/dt = d \log D_t^*/dt - d \log N_t/dt = g^* - n , \qquad (3)$$

so that

$$D_t = xY_t = xY_0 e^{(g^*-n)t} = xY_0 e^{gt} = D_0 e^{gt} . \qquad (4)$$

Since the sum of current cash returns (here dividend yields $= D_t/P_t = y_d$) plus rates of growth in own price for all assets must equal the current market rate of discount in equilibrium in perfect markets, the basic *equilibrium price condition* is $y_d + d \log P_t/dt = k_t$. The solution of this differential equation for market price, recognizing (4) and letting x be constant for simplicity, is

$$P_t = \frac{D_t}{k-g} = \frac{D_0 e^{gt}}{k-g} = \frac{D_0 e^{(g^*-n)t}}{k-(g^*-n)} = \int_t^{\infty} D_{te}^{-(\tau-t)(k-g)} d\tau , \; k > g, \qquad (5)$$

14. These latter stipulations incorporate the economist's usual (and seemingly very realistic) assumption that marginal rates of return on investment budgets are not infinitely elastic *as of any given point of time* throughout most of their relevant range; that they become so only with respect to outside investments in the market after all internal investments having higher marginal returns have been exhausted. Cf. Duesenberry [a6].

which, by derivation, will satisfy the criterion cross-sectionally over different stocks and securities *and* will do so continuously over time.[15] For the *current* price of the stock, P_0, equation (5) reduces to

$$P_0 = \frac{D_0}{k-g} = \frac{xY_0}{k-g} = \frac{xY_0}{k-(g^*-n)}, \quad k > g. \tag{5a}$$

Since, in the absence of issue costs, taxes, and uncertainty, all forms of financing are perfect substitutes at the margin, the marginal costs of each are the same. The minimum acceptable marginal rate of return to justify any additional internal investment under these conditions, however financed, can most easily be found for retained earnings. With $n = 0$ and $g^* = g$, and letting the retention ratio $r = 1 - x$,

$$\frac{\delta P_0}{\delta r} = P_0\left[-\frac{1}{x} + \frac{\delta g/\delta r}{k-g}\right] \gtrless 0 \text{ as } \frac{\delta g}{\delta r} = \rho \gtrless \frac{k-g}{x} = Y_0/P_0 = y_e, \tag{6}$$

where y_e is the current earnings yield on the stock. The marginal internal rate of return $\rho = \delta g/\delta r$ in this case because $\delta f/\delta r = 1$ and[16] $\rho = \delta fp(f)/\delta f = \delta g/\delta r$.

Under fully idealized neoclassical conditions with opportunities for constant growth forever, the optimizing decision rule is to accept all investments having $\rho \geq y_e$, the current earnings yield.[17] But y_e

15. In [*b*8] we give the more general form of this model in which dividend payouts (earnings) growth rates, rates of issuing new securities, and discount rates are all unique functions, each varying in any way over time, and show that the resulting model also has the properties just stated in the text. A *corollary* of critical importance is that any alternative model of stock prices (such as various models based on earnings) will satisfy this criterion of legitimacy in classical theory *if and only if* it is *identically reducible* to the dividend model. (For further elaboration see Lintner [*b*7].) We consequently do not need to use any such alternatives to dividend models in this paper.

16. Alternatively, by definition

$$p(f) = \frac{1}{f}\int_0^f \rho(f)\,df \text{ or } g = \int_0^f \rho(f)\,df,$$

and the text relation follows by direct differentiation.

17. This is precisely the rule advanced by Modigliani-Miller in [*a*23], which was derived from a "corporate earnings" model and under essentially static assumptions; but, as noted in Lintner [*b*7], for use in dynamic situations their original definition of "current earnings" must be altered to the more traditional concept in which *current* rates of earnings flows (rather than the undiscounted time-average they proposed) are used directly in the numerator of the relevant *current* earnings yield. It is a nice paradox that our model basing values on dividend flows in the steady-growth case under certainty leads to an optimizing rule based on straight *current* earnings yield—which had been advocated by most earnings theorists all along on the basis of a price model which is *not* generally valid in dynamic contexts! (see Lintner [*b*7, p. 249]); that the "market rate" used to discount dividends in these models is seldom the correct cutoff rate; and that the equation of earnings yields to market "discount" rates often presumed holds up in growth situations only on very restrictive additional conditions on profit opportunities.

is equal to the discount rate k *only* in the special case where profit opportunities are infinitely elastic throughout—i.e., strictly constant returns regardless of the size of the investment budget in each period—and at a level equal to k.[18] This establishes the second half of the proposition made at the beginning of this section. The extremely unrealistic character of these conditions indicates that the common assertion that optimal investment budgets can be set by equating the corporation's ρ to k is generally in error, even under otherwise idealized conditions, when steady growth is assumed.[19]

Indeed, if the company is operating in the region of diminishing returns so that $\rho < p$—and this is surely the usual case—then $\rho_0 < k$ so long as the company is paying any dividends:[20] the minimum marginal returns *to the company* which will lead investors under the conditions being assumed to prefer added company investment is necessarily (and often very significantly) *less* than the discount rate k, which *inter alia* reflects returns available on alternative investments.[21] The explanation is that from equation (5) $\delta P/\delta r > 0$ as $x\rho + g \geq k$. The marginal return for the *investor* from added investment within the company is equal to the sum of the dividend payout applied to the marginal internal return within the company *plus* the growth rate on the retention itself, and if this *sum* is greater than the discount rate, he will prefer the retention.[22]

It must be also emphasized that the marginal cost of capital (m.c.c.) is the *current* earnings yield $y_e = Y_0/P_0$, *not* the ratio of future or "average future" earnings to current price, as frequently proposed [e.g., in *a*30, *a*23, and *a*26]). Moreover, the earnings yield

18. Since $y_e = (k - g)/x = [k - (1 - x)p]/x$, we have $xy_e + (1 - x)p = k$, so that $y_e \gtreqless k$ as $p \lesseqgtr k$, since $0 \leqslant x \leqslant 1$. But $p < k$ can, of course, be ruled out in any well-managed corporation, and we are left with $y_e \leqslant k$ as $p \geqslant k$.

19. This error in Gordon and Shapiro's conclusion to this effect [*a*14] has been noted by Bodenhorn [*a*2]. The still more recent paper of Modigliani and Miller [*a*25], however, continues to use the discount rate k as the cost of capital in the "steady growth" case.

20. From eq. (6) we have $x\rho_0 = k - (1 - x)p$ or $k = p - (p - \rho_0)x$. If $x > 0$ and $\rho_0 < p$, then $p > k$, and the conclusion follows from the second preceding footnote, since $\rho_0 = y_e$.

21. This is, of course, contrary to the case treated above, where profit opportunities were independent of investment rates in other periods. The reason for the perhaps surprising conclusion that $\rho_0' < k$ clearly lies in the different assumptions on investment opportunities.

22. It will be noted that *this result does not depend upon any tax differentials* between ordinary income and capital gains rates such as have been so much emphasized in the literature. The broader significance of the result is to emphasize the importance of the distinction between marginal return requirements *to the company* and marginal returns to investors; for even though $\rho_0 < k$, the investor's return given above *does* meet the opportunity costs of returns on alternative investments reflected in k.

declines with increasing size of capital budget *up to* the optimum scale of investment, since with $y_e = (k - g)/x$, $\delta y_e/\delta r = (y_e - \rho)/x < = > 0$ as $\rho > =$ or $< y_e$: *the act of making appropriate company investments reduces* y_e [= m.c.c. under present assumptions] and does *not* raise it as alleged elsewhere (e.g., [*a*30]); *only improper investment raises* y_e.

Before turning to uncertainty, I should also show that the internal returns required to justify expansion financed externally in the face of underpricing and new issue costs are substantially greater than so far recognized. The basic valuation model is still equation (5), but with new stock issues the aggregate size of the capital budget now is $F_t^* = Y_t^* - D_t^* + S_t^*$, where S_t^*, is the *net* dollar proceeds to the company from any newly issued shares. Dividing through by Y_t^*, we now have $f = r + s$ for use in equations (1) and (1*a*). Differentiating equation (5) partially with respect to s gives

$$\frac{\delta P_0}{\delta s} = -\frac{P_0\left[\frac{dn}{ds} - \frac{\delta g^*}{\delta s}\right]}{k - g^* + n} \gtreqless 0 \text{ as } \frac{\delta g^*}{\delta s} = \frac{\delta g^*}{\delta f} = \rho \gtreqless \frac{dn}{ds}. \qquad (7)$$

To relate s to n, the relative rate of issuing new shares, first note that, in the absence of issue costs, net proceeds to the company are equal to the price to the buyers and also that, under classical certainty, the aggregate market demand for the company's equity shares is infinitely elastic at the initial (pre-new issue) price earnings ratio, y_e.[23] Under *these* conditions,[24] $s = n/y_e$ and $dn/ds = y_e$, so that required returns for "costless" new equity financing are the same as those found above for retained earnings.[25]

23. This is true under these conditions because aggregate market value (in the absence of debt) is independent of number of shares, so that *both* price per share and earnings per share are rectangular hyperbolas in terms of number of shares, and the ratio is constant at y_e. This formulation has also been used by Kuh in [*a*16].

24. Where P_t^* is the aggregate market value of the stock in the absence of new issues, and N_t the total number of shares outstanding at time t, the price per share P_t in the absence of new issue costs is determined by $N_t P_t = P_t^*$, where P_t^* is a constant independent of $\Delta N_t = dN_t/dt$ the number of new shares issued in the given time interval. Also $N_t = N_{t0} + \Delta N_t$, where N_{t0} is the number of shares in the absence of new issues; S_t^*, the aggregate net proceeds of the new share issues, will then be

$$S_t^* = \int_{N_{t0}}^{N_{t0}+\Delta N_t} \frac{P_t^*}{N}\, dN = P_t^* \int_0^n dn = P_t^* n\,.$$

Consequently, $s_t = S_t^*/Y_t^* = P_t^* n/Y_t^* = n/y_e$.

25. The equivalence of required returns when retentions or new stock issues are used *under these conditions* to finance expansion—and hence the indifference of shareholders between more dividends cum more new issues vs. more retentions cum smaller new issues (and so a larger percentage of ownership represented by given initial share holdings)—can also be confirmed by showing that the total differential of P_0 in eq. (12*a*) with f (and consequently g^*) fixed is equal to zero.

But in real life there are both fixed and variable costs of issuing new equity securities and, in addition, some "sweetening" in the form of pricing under the current market is usually required to sell new securities.[26] Such overt costs and underpricing can be summarized by making the (average) *net proceeds per share* on the new issue be $p_{t0}\ (a - bn) - c$,[27] so that[28] $s = [n/y_{e0}]\ (a - bn) - c$, $0 < a \leq 1, 0 < b < 1, n \geq 0, c > 0$. With these costs recognized $dn/ds = y_{e0}/(a - 2bn)$, and $\delta P_0/\delta s >$ or $= 0$ only so long as $\rho >$ or $= y_{e0}/(a - 2bn)$, where $t_{e0} = Y_t/P_{t0}$ as defined above. The proper cutoff on new stock issues (even in "growth" situations) is the ratio of the *current* (not future) earnings to the *marginal* net proceeds per share $P_0\ (a - 2bn)$ of the new stock—and not simply P_0a as commonly proposed (e.g., in [a30]). Moreover, since the return required to justify expansion financed by stock issues in the presence of any unavoidable "underpricing" and of any overt issue costs is greater than that required to finance expansion by retained earnings, there is a vertical shift in the "supply or cost of capital function." Investors will consequently always prefer, in the context of the present model, that investment budgets be financed with retained earnings instead of new stock issues as long as retained earnings are available—i.e., so long as $x > 0$ and $r < 1$. Companies optimizing for shareholders, however, *should* expand capital budgets further by issuing new shares after retentions are exhausted, so long as the stated marginal condition can be satisfied. Finally, it is apparent that the absence of current dividends does not nullify the applicability of our present model based explicitly on dividend flows: the value of *currently* outstanding stock is still simply the present value of the dividends which will be paid in the future on the presently outstanding shares.[29]

26. Strictly speaking, "underpricing" would never be required in classical markets under certainty, but I have shown in [b7] that (*a*) it is unavoidable under uncertainty whenever diverse probability distributions over outcomes is admitted and (*b*) its impact is essentially the same (though different perhaps in degree) as fixed and variable costs under certainty. To save space in the present exposition the two have been treated together at this point.

27. The subscript zero refers to values that would have obtained if n were zero; $(1 - a)$ represents the minimum fractional underpricing required to sell *any* new stock, while b covers both the variable cash costs of issuing new securities *and* the *further* underpricing which is dependent on the size of the new issue—the units of both a and b being fractions of P_{t0} as defined; c denotes the *fixed* overt cash cost per share of new issues.

28. With new issue costs recognized, using symbols defined just above, *aggregate* net proceeds to the company are $S_t^* = P_{t0}^* n(a - bn) - c$; the equation given follows after dividing by Y_t^* when $y_{e0} = Y_t^*/P_{t0}^*$. It should be noted that uncertainties concerning the b or c will further increase the marginal cost of new outside equity relative to that of retained earnings.

29. Both the latter two points can be nicely illustrated by considering the simple case of a company whose investment opportunities over a period of m years will be so

III. Firms under Uncertainty

I now turn to some important general conclusions required by the fact of uncertainty. The first is that, as I pointed out two years ago,[30] while decision rules for determining the optimal size and project composition of capital budgets are generally identical under neoclassical certainty, *they are essentially different under uncertainty: the problem of optimizing the composition of a capital budget of any given size is formally identical with problems of selecting optimal security portfolios.*[31] In the rest of this paper, I shall focus on the optimal determination of the size of the capital budget and the mix of internal funds and debt to be used in its financing, simply assuming that a Markowitz-type "efficient set" analysis has already been made which yields a three-dimensional (per time period) "profit possibility function" relating amount of investment (size of budget), expected average profit rates, and variance of return.

In keeping with our emphasis here on (expectationally) steady growth, however, I assume specifically that $\psi_t(f, \hat{p}, \hat{\sigma}_p^2) = 0$ is invariant over time with $f = F_t^*/Y_t^*$ constant at some level to be determined, and that $\delta\hat{p}/\delta f \leq 0$, which is constant over time for any f and σ_p^2 as is the marginal expected rate of return $\rho_\wedge = \delta f\hat{p}/\delta f$. Also, to simplify the development and concentrate on the budget-portfolio returns required in the presence of given company-investment risks, I will assume that the profit-rate-variance σ_p^2 of the budget is fixed or prespecified, that it is invariant over time, and that *it* is also invariant to the size of the budget. Since $\sigma_p^2 > 0$ (even when $f = 0$, so that $\hat{g} = 0$), however, and since $g = fp(f)$, in general the variance of the growth rate $\sigma_g^2 = (1 + a_1f^2)\ \sigma_p^2$, which does depend on size of

rich that no dividends should be paid during this time, after which its special investment opportunities will be gone and it will pay all subsequent (constant) earnings in dividends. The present value of the stock will be $P_0 = e^{-km}\ Y_m/k$ and $Y_m = Y_0e^{(g^*-n)m}$. Maximizing P_0 involves maximizing Y_m, which leads immediately to the optimizing rule given above.

30. [a18]; the final page of Hirshleifer's paper [b6] at the same meetings makes the same point. See also my [b8] and [b9].

31. As a result, individual investments (projects) may be eminently desirable components of optimizing project-portfolio-budgets because of low variances and/or covariances with other components (and existing assets) in spite of relatively low expected returns. Also, a project having a large variance in its own quasi-rents but low or negative covariances with other existing and future investments will often make a much smaller contribution to company-wide variance (risk) than other projects with low own-variances and substantial intercorrelations with other company investments in terms of cash flows. Indeed, many investment proposals are accepted in capital budgets in order to reduce risks and not to raise returns—something incongruous in conventional theoretical contexts of capital budgeting, but surely to be expected in the present framework. It should be clear that throughout I take p and σ_p^2 to refer to profit *before* interest.

budget although invariant over time for given f. Cumulated growth over a period t in length, $\bar{g}t$ is then a random variable with $\sigma^2_{(\bar{g}t)} = t\sigma^2_g$.

In what follows, I examine certain important properties of the *comparative stochastic dynamics* of capital budgeting, corporate financing, and growth, seeking the marginal cost of capital (as previously defined), and the decision rules for the optimal determination of f (and its components, relative rates of retentions r and borrowing θ) on the assumption (management's and investor's expectation) that the values of f, r, and θ decided upon will be held constant over time.

With the current price of the stock now a random variable, our *criterion* becomes maximization of the expected value of this current market price, and, to save space, I shall here simply assume[32] that this expected value is equal to the *present* value (computed essentially at the risk-free discount rate[33]) of the *certainty equivalents* of the uncertain income (dividend) receipts in the stream. I also assume that, at the time of the company decision (i.e., on pre-existing data and expectations), all investors hold the portfolios they most prefer. Any change in the retention ratio (dividend payout), leverage, or expected growth rate of the ith company which increases the present value of its stock will increase its shareholders' wealth and be in their interest.

Our problem essentially involves the terms of trade between expected receipts and varying risks on a given security—"deepening" in Hirschleifer's terminology [*b*6] rather than (or along with) the much simpler "widening" case he examined. It is clear, however, that the functional relation between certainty-equivalents, expected returns, and risks must fall between two limiting cases.

On the one hand, in the limit, under the extreme simplifying as-

32. Some of the deductive justification is given in Hirschleifer [*b*6] and Smith [*b*11]. See also my [*b*8]. For those who prefer to use the alternative criterion of the certainty-equivalent of the probability distribution of the present values of the uncertain streams, I will simply observe that, under a very general set of assumptions otherwise, all the general conclusions drawn below also hold under this alternative criterion where it is viable.

33. To a reasonably good approximation, these present values can be computed at the risk-free discount rate k_0 for the average or representative company. Provisional present values of all securities computed with discount rates $k_i = k_0$ may, however, lead to switching and other adjustments, which results in changes in (expected) market prices. When all portfolios are in full adjustment on the basis of a given set of underlying expectations, parameter values, supplies of securities, etc., the expected price of any ith security can be equated to the present value of its certainty equivalents computed at a discount rate $k_i = k_0 + k_{ci}$, where k_{ci} (either + or −) reflects the impact of changes in share price P_i due to switches, covariances, etc. In the text I drop subscripts and implicitly assume k_{ci} to be invariant, but, in general, k_{ci} will vary with σ and compound to results stated below.

sumption that all trades are between single risk assets (or portfolios of fixed proportions) and *riskless* securities, we know that all investors' marginal rates of substitution are equal in equilibrium to market-determined exchange lines which are linear in expected return and σ as a measure of risk. But continuing to assume purely competitive markets (as I do throughout), the exchange lines governing expected prices for given expected returns (or expected returns required for given prices) *within the set of risk assets* and when "money illusion" is absent, involve both σ and σ^2. The second limiting case is provided by the observation that market equilibrium with interior solutions requires that the marginal rate of substitution on the latter function not exceed that on investors' utility functions—and in the absence of good viable markets for trading in the relevant disjoint future uncertain receipts, the latter must in themselves provide the certainty equivalents. (See my [b8] and [b9]).

Consider now the second limiting case, letting investors' utility functions, following Tinbergen [b14],[34] be hyperbolic of the form $U(\tilde{D}_t) = 1 - (C_0/D_t)^{a_1}$, $\alpha > 0$. Then[35] $E[U(\tilde{D}_t)] = 1 - (C_0/\check{D}_t)^{a_1} = 1 - (C_0/\check{D}_t)^{a_1} = 1 - (C_0/D_0)^{a_1} e^{-a_1 t(\hat{g} - a_1\sigma_g^2/2)}$ so that the certainty equivalent is $\check{D}_t = D_0 e^{t(\hat{g} - a_1\sigma_g^2/2)}$, from which the stock price is

$$P_0 = \int_0^\infty D_0 e^{-t[k - \hat{g} + a_1\sigma_g^2/2]} dt = \frac{D_0}{k - \hat{g} + a_1\sigma_g^2/2}, \quad k + a_1\sigma_g^2/2 > \hat{g}. \quad (8)$$

The impact of uncertainty can be clearly seen in the marginal cost of funds for internally financed expansion, which is

$$\frac{\delta P_0}{\delta r} = P_0\left[-\frac{1}{x} + \frac{\rho_\wedge - a_1(\delta\sigma_g^2/\delta r)/2}{k - \hat{g} + a_1\sigma_g^2/2}\right] \quad (9)$$

$$> 0 \text{ as } \rho_\wedge > y_e + a_1 a_1 f \sigma_p^2 = \text{m.c.c.}$$

Uncertainty, of course, raises the earnings yield, but the more subtle and far-reaching result is that, *in addition, the marginal cost of capital* (here retained earnings) *is greater than the earnings yield by amounts which vary directly with size of the capital budget f* and the size of the coefficient a_1 in $\sigma_p^2 = (1 + agf^2)\,\sigma_p^2$. (Note also that this result was reached even though the marginal profit variance σ_p^2 on the

34. Quadratic utility functions, however, are patently inappropriate in the context of our concern with long-run growth, even if variances are minimal or zero. A point is soon reached beyond which further increases in dividend (and growth) would *reduce* the utility of the receipt. The hyperbolic form adopted here is free of this disability and has other important advantages [cf. b8].

35. Cf. Aitchison and Brown [b1, p. 8].

capital budget itself was assumed constant. If σ_p^2 also varies with f, the result is compounded.) Moreover, I have shown elsewhere [*b*8] that these results are quite general. In particular, if viable markets for all future time periods exist which establish exchange lines along which $\hat{g} - \alpha_2\ \sigma_g - \alpha_3\ \sigma_g^2$ are equally valued (the first limiting case above is covered by setting $\alpha_3 = 0$), these same conclusions hold, with the excess of m.c.c. over y_e varying directly with the market exchange coefficients α_2 and α_3 instead of directly with α_1, the coefficient of risk aversion on the utility function itself. Finally (as also shown in [*b*8]), m.c.c. $> y_e$ necessarily, and by amounts that increase essentially exponentially with the size of budget, if when viewed as of t_0, the variance $\sigma_{p_t}^2$ of the profitability of new investments to be made at different times in the future is a monotone increasing function of their futurity. With this very plausible and persuasive feature incorporated in the models, the conclusions stated above hold *even if*[36] $a_1 = 0$.

These results lead directly to other fundamental conclusions. Even though leverage per se has not yet been considered explicitly, it necessarily follows from the preceding analysis that *the conventional weighted-average-cost-of-capital rule is inherently erroneous and down-biased. Even if* a weighted average of equity and debt costs were the proper criterion, the average of earnings yield and interest cost would be *too low* because the relevant marginal cost of retained earnings is *greater* than the earnings yield (and the relevant marginal cost of outside equity still larger). If, for instance, both retained earnings and debt are to be used in financing, standard production theory insures that (a) the optimal *mix* will involve the *equalization of the two* (interdependent) *marginal costs* and (b) the relevant marginal cost of (optimal-mix) finance for any sized budget will be *equal to the* (*equalized*) *marginal costs of each type of finance used.*

Even with quoted interest rates well below equity yields, there is, of course, no problem in having marginal costs of debt equal to marginal equity costs: not only are marginal interest costs with much use of debt substantially above stated or coupon rates, but—just as non-zero profit-rate variances make the *relevant* marginal costs of equity greater than earnings yields—it is reasonable to expect that the *relevant marginal costs of debt will similarly be greater than*

36. The reason is essentially that increased retentions and growth shift relatively more of the income stream into the further future and thereby increase the relevant weighted average uncertainty of the stream.

even the marginal overt interest costs. And so they are.[37] Although borrowing per se does not affect σ_p^2, the variance of the profit rate before interest, by introducing fixed interest charges it necessarily increases $\sigma_{p_a}^2$, the profit rate variance after interest, and consequently σ_g^2, which is the variance more directly relevant to the shareholder.[38] Moreover, it does so at every point in time and cumulatively over time—and as interest costs increase with increased borrowing, it does so in necessarily non-linear fashion even on the standard deviation and a fortiori so on the variance. Such (non-linearly) increasing shareholder risks with increasing corporate borrowing raise the relevant marginal costs of debt (minimum expected marginal returns on investments) *above* its marginal overt interest cost (which is its true marginal cost under certainty)—and by margins which progressively increase with the relative amount of the debt financing—for precisely the same economic reason that any increase in risks in the shareholders' income stream due to added retentions raises *their* true marginal cost above the earnings yield (which would have been their proper marginal cost under certainty).

IV. Conclusion

In conclusion, it should be emphasized that *so long as* the marginal expected return on the capital budget is > *this* m.c.c. of debt (making full allowance for its risk impact), *debt-financing-cum-investment raises* the (expected value of) the *current stock price*—and consequently *lowers current earnings yields,* contrary to the common impression. *Only unjustified debt-financed-expansion raises current earnings yields.* Of course, $\rho_\wedge <$ m.c.c. (debt) *until* r (or s) is substantially positive; but in these models, *after* r and s have been optimized under the constraint of no (permanent)[39] borrowing $\rho_\wedge$ will often be > m.c.c. (debt) and permanent borrowing is desirable (because it raises share values) up to a well-defined optimum,[40] again contrary to theoretical models now current; alter-

37. This analysis is free of the straitjacket of the "entity value" theory for reasons given in detail in [b7].

38. If $a_1 > 0$, as is surely the usual case, borrowing increases this variance in compound and non-linear fashion [since $(1 + a_1 f^2)\ \sigma_{pa}^2$ is a product]. With borrowing in the picture, $f = r + \theta$, where θ is the new borrowing and all variables as before as ratios to current earnings.

39. In view of the emphasis on comparative dynamics, θ is defined as a fraction of earnings, and, with positive growth, total debt grows continuously over time as in the Domar models. Our θ does not include temporary borrowing to even out stochastic variations in income flows.

40. After borrowing is optimized subject to r (or s) fixed at *its* optimum assuming no debt, further retentions will often *become* justified (due to interaction effects between costs of equity and debt capital) and so on interatively to the global optimum.

natively, so long as the equity financing exceeds a certain pace, there is an optimal finance-mix involving both equity and debt for each relative size of budget f, and along this finance-mix "expansion path," budget size f should be increased until the condition $\rho_\wedge \geq$ m.c.c. is no longer satisfied.[41]

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41. Depending on parameter values, it is entirely possible (and probably frequent in practice) that $\rho_\wedge <$ m.c.c. (debt) for *all* values of r (and s), in which case the optimum borrowing $\theta = 0$ throughout.

OPTIMAL DIVIDENDS AND CORPORATE GROWTH UNDER UNCERTAINTY *

JOHN LINTNER

Three decades ago, Berle and Means emphasized the separation of ownership and management in the typical modern corporation.[1] I have elsewhere examined whether some of the dire and gross adverse implications foreseen from this development have been borne out by subsequent experience.[2] In the present article, I shall focus on the separate question of *how* a distinct professional management "should" determine some central investment and financing policies *if*, in keeping with traditional presumptions (and classical prescriptions), it were to seek to make these decisions in ways which would be in the best interests of their common shareholders.

The need for a careful examination of this problem is highlighted by the fact that the very proliferation of writings on this issue over the last decade has resulted in a spate of inconsistent and mutually contradictory prescriptions.[3] In addition, most of these authors have rested their analysis on various proximate, simplifying assumptions such as prescience, stock prices equal to simple capitalizations of company earnings, static conditions, and so on. In keeping with these other authors, however, I shall identify optimization in terms of shareholder preferences with the maximization

* This paper is one part of a series of interrelated theoretical and statistical studies of corporate and financial policies being made at the Harvard Business School under a grant of the Rockefeller Foundation for work in the general area of profits in the functioning of the economy. The Foundation's generous support for this work is most gratefully acknowledged.

Note: At various points in the paper, reference is made to appendix notes. These notes provide rigorous mathematical proofs of various propositions and properties stated in the text, and are available in mimeographed form upon request to the author. For convenience, the fifth note, for instance, will be designated [A5].

1. Adolph A. Berle and Gardner C. Means, *The Modern Corporation and Private Property* (New York: Commerce Clearing House, 1932).

2. John Lintner, "The Financing of Corporations," Chap. 9 in *The Corporation in Modern Society*, ed. E. S. Mason (Cambridge: Harvard University Press, 1959).

3. For a summary of these different conclusions and decision rules, see the first part of John Lintner, "The Cost of Capital and Optimal Financing of Corporate Growth," *Journal of Finance*, XVIII (May 1963).

of the current market price of the common stock (since increased share values increase current wealth and thereby utility). But since there is "noise" in current market prices when uncertainty is present (even given all relevant data), and since speculation on *general* stock market price moves is irrelevant to the major decisions studied in this paper, my criterion becomes the *expectation* of current equity value, *given* (or relative to) the level of (say) the Standard and Poor's or Dow Jones Index — i.e., $E(\tilde{P}_0/SP \ldots.)$ — and I assume that the corporate decisions of interest are (or "should be") made to maximize this value.

There has, of course, also been a substantial literature examining the goals and objectives the "managerial enterprise" seems to be seeking — that part most relevant to the present article either assuming or arguing that management should seek to maximize growth in assets or sales, perhaps subject to a constraint on profits, or insecurity, or both. My concern at this time is not to argue the merits of these generalizations of what management *is* doing, nor even to argue that it should seek exclusively to serve the shareholders' interests; it is rather to develop some of the implications of this alternative and more traditional standard, in part so that any contrasts in implied behavior can be seen in clearer focus.

To this end, I shall advance a model of corporate growth and equity values under dynamic but inherently stochastic conditions, and use this model to determine (decision rules for) the optimal size of capital budgets, dividends, retentions and expected rates of growth over time. The growth model used is quite comparable at the micro level in spirit to the classic macro-growth models of Domar, Harrod, Solow, and Tobin, having as its core a "profit possibility" function which subsumes much suboptimizing behavior with respect to product lines, markets, channels, pricing, promotion, and decisions on the composition (or internal project-mix) of the capital budget — just as the macro-prototypes are built around a master production function which subsumes a great deal indeed. But in comparison with these prototypes, our model is generalized in three fundamental respects: (1) its "profit-possibility" function exhibits diminishing returns *as of* any given point in time — and thus is not restricted to (the equivalents or implications of) either constant costs or returns; (2) it explicitly incorporates specific stochastic processes over time whose parameter values are (functions of) decision variables within the firm, thereby focusing upon problems of optimization under uncertainty rather than under the blissful prescience of most of its macro prototypes; and this in turn requires (3) explicit use of von Neumann-Morgenstern type preference or utility functions, or market opportunity lines between expected returns and risks (which, in general, depend *inter alia* upon the form and parameter values of these utility functions). In keeping with the modern emphasis upon disaggregated micro "moving parts" in macro models, it is hoped the present analysis will contribute insights and subassemblies useful in subsequent macro analysis, but the concern of this paper will remain at the micro level throughout.

In more detail, I examine equity values and capital budgets and their financing under inherently stochastic conditions which have the property that *expected values* of corporate earnings, stocks of capital invested and market prices of the corporate equity follow exponential growth trends whose underlying parameters include variables subject to decision by the firm. In particular the relative size of the investment budget, the marginal expected rate of return on this budget (and the variance of this rate of return), along with the dividend payout and retention ratio (and in the general case relative rates of outside equity and debt financing) are all functionally interrelated variables and subject to decision. In order to keep the present paper within reasonable bounds, however, and to sharpen the focus on decision rules for optimal size of capital budgets and rates of growth — and specifically upon dividends and retentions as a source of financing, in view of their central role in business financing as well as in the general Berle-Means position — the analysis at this time will be confined to the case of firms growing entirely through retained earnings. In addition, the analysis here is confined to firms growing under conditions in which the variance of *profit rates* is a predetermined variable [4] independent of the size of capital budgets (although the marginal expected profit rate is a declining function of budget size at any given time). Our concern in this paper is thus, in Hirschleifer's terminology,[5] with issues of optimal

4. In order to generalize along the other important dimensions indicated, we thus abstract from the issues raised by the determination of the optimal *mix* of financial inputs (retained earnings, new stock issues and debt) into our "profit-production function," and from the closely related question of the optimal degree of deliberate risk bearing for the firm. Some of these issues excluded here have already been examined in John Lintner, "Dividends, Earnings, Leverage, Stock Prices and the Supply of Capital to Corporations," *Review of Economics and Statistics*, XLIV (Aug. 1962), and "The Cost of Capital and Optimal Financing of Corporate Growth," *op. cit.* Others are developed more fully, using the basic models of the present paper in the author's "Optimal Risk Bearing, Retentions, and Leverage in Corporate Growth," forthcoming.

5. Jack Hirschleifer, "Risk, the Discount Rate, and Investment Decisions," *American Economic Review*, LI (May 1961), 113–14.

risk-*widening* through capital budgeting decisions, rather than with risk-deepening.

Moreover, the essential focus of the analysis (like that of the classic macro-growth models) is upon *comparative dynamics* (analogous to classical comparative statics) rather than upon period-by-period decision-making after the manner of dynamic programming. Not only is the comparative dynamics analysis more tractable for our present issues but it is also very relevant: extensive empirical work has shown that a wide range of individual decisions of corporate managements on matters of finance and investment are typically influenced strongly by basic guidelines or *target* values of dividend payouts, debt-equity ratios, relative size and profitability of capital budgets and so on,[6] which essentially reflect long-run rather than relatively short-run or transient considerations and objectives. The objective of the present paper concerns the question *how* these more stable levels should be determined if they are to be selected in the shareholder's interests. In keping with this essential focus, I shall simply assume that the expected values of market prices are equal to (or a monotone increasing function of) present values computed *as if* the *relative* sizes of capital budgets, finance-mix, etc., currently chosen will be maintained over time.[7] Also, of

6. See John Lintner, "Effect of Corporate Taxation on Real Investment," *American Economic Review*, XLIV (May 1954), and "Distribution of Incomes of Corporations Among Dividends, Retained Earnings and Taxes," *American Economic Review*, XLVI (May 1956); Gordon Donaldson, *Corporate Debt Capacity* (Boston: Division of Research, Graduate School of Business Administration, Harvard University, 1961); John Meyer and Edwin Kuh, *The Investment Decision* (Cambridge: Harvard University Press, 1957); Myron J. Gordon, "Security and a Financial Theory of Investment," this *Journal*, LXXIV (Aug. 1960); Joel Dean, *Capital Budgeting* (New York: Columbia University Press, 1951); W. A. Locke Anderson, "Corporation Finance and Fixed Investment: An Econometric Study," mimeo.; as well as any of the better texts on corporation finance. There is also a substantial body of evidence that such longer-run targets (and successive shorter-run partial adaptation to target ratios) of market shares, gross margins, etc., are important in pricing and merchandising, and that similar considerations are important in wage settlements.

7. We do *not* assume that there is a presumption on the part of either managements *or* investors that these decision parameters (or their underlying determinants) will not in fact change over time. We *do* assume that these underlying determinants (notably profit rates as a function of relative size of budget) *expectationally* are stochastic processes, and that the decision parameters (a) are based upon the expectations *and* variances of these underlying stochastic processes, and (b) that these (long-run) decision parameters will not be changed frequently or in the short run. Considering the force of discounting over several years, the effects of well-deferred future changes in decision parameters on present values is substantially muted, and their effect upon the current choice of appropriate parameter-values is still more attenuated. As it stands, therefore, our analysis yields good first order approximations to true optima, ignoring only second or third order effects. Moreover, since the effects of such deferred future changes in decision parameters would be to capitalize on cumulatively very favorable developments or minimize the

course, I assume throughout for purposes of this theoretical analysis that maximizing behavior is universal and that all financial markets are purely competitive. In particular, I assume that each investor in the market holds that portfolio — including stocks, bonds, and other investments, real estate, etc. — which he most prefers, and examine the effects of i^{th} company decisions on the price of *its* stock. For simplicity, I also assume that all tax rates are zero, and that the (riskless) discount rate $k_{\tau} = k$ is constant over time.

Section I develops certain important concepts and essential elements of our analytical model. Section II completes the model under conditions of certainty and briefly establishes certain of its properties under these simple prescient conditions. Although extremely unrealistic, rigorous examination of this limiting case provides important benchmarks and inputs to the more general analysis under uncertainty in the rest of the paper. In particular, it is shown that — just as profit maximization in the standard "theory of the firm" implies equality between marginal revenues and marginal costs — maximizing equity values implies corresponding equality between marginal (expected) rates of return and an appropriately derived "marginal cost of capital," and the shape of the latter function is examined. Section III introduces uncertainty and examines its impact on decisions and decision rules under the simplest possible stochastic conditions, develops and uses a "present value of certainty equivalents" [8] (with discounting at a risk-free rate) as the model of stock values, and again derives appropriate marginal conditions for the optimum. Section IV then develops the analysis under more realistic assumptions regarding the underlying judgmental stochastic process.

I. Some Important Definitions and Building-Blocks

In this paper, I shall consistently use decision rules in the form of marginal internal rate of return vs. marginal cost of capital. This is done simply as a matter of convenience. Under the assumptions made concerning the efficient set of the (portfolio) of investment opportunities facing the firm (and their expectational stability over time) these rules are strictly equivalent to the alternative statement of rules in the form of present values exceeding costs. This is

adverse impact of unfavorable developments, the prospect of such future changes in decision parameters can be adequately handled within the framework of the present analysis by a moderate adjustment in the risk parameters introduced below.

8. It should be noted, however, that the certainty equivalents are functions, *inter alia,* of *both* variances and covariances.

true because the *marginal* (expected) *rate of return* schedule plotted vertically against the relevant size of the capital budget on the abscissa simply indicates, for each size of budget, the *maximum rate of discount* (assumed constant over time) which can be used in computing the (expected) [9] *present values* of the (expected) differences in the company's cash flows attributable to the presence or absence of each separate candidate project in the capital budget, and still satisfy *two* constraints: (a) the total capital budget must be of the indicated size when (b) it includes *only* those potential projects which satisfy the standard present value criterion [1] in Lutz's notation, $V \geqslant C$.[2]

Corresponding to this definition of *marginal* (expected) rate of return — which for convenience has been stated in general form to cover uncertainty (certainty being the limiting case for each when all variances approach zero) — we define the *marginal cost of (a given type of) capital* as the *minimum* (expectation of) *rate of return required on a marginal investment* in the current period — or, equivalently, the *minimum marginal expectation of rate of return* on the entire capital budget — *for the shareholders to be better off* (value of existing equity greater) *with* the incremental-investment-cum-this-incremental-financing *than without either* the increment to the capital budget *or* this financing.

9. It will be recalled that for given interest rates, expected present values of stochastic flows are equal to the discounted sum or integral of the expectations of the marginal (statistical sense) distributions of receipts at each point in time even when there is time-interdependence in the receipts.

1. In addition, of course, where there are mutually exclusive candidate projects, only the one with the greatest present value at any given stated discount rate will be included in the budget for that rate. Also, when two or more separate projects are interdependent, each possible combination of component projects should be entered as a separate potential project. Note that our assumption of constancy over time in k at a level *fixed independently* of the firm's "investment demand" or "aggregate marginal efficiency of capital schedule" obviates the problems raised by Frederick and Vera Lutz, *The Theory of Investment of the Firm* (Princeton: Princeton University Press, 1951), pp. 155–62. It will be noted that the rule stated in the text also cuts through the problems raised by any multiplicity which may be present in rates of return on individual projects (cf. James H. Lorie and Leonard J. Savage, "Three Problems in Rationing Capital," *Journal of Business*, XXVIII (Oct. 1955), since it identifies a unique "opportunity value of funds" for the budget and hence for the decision on an individual project. Cf. Ezra Solomon, "The Arithmetic of Capital-Budgeting Decisions," *Journal of Business*, XXIX (April 1956), reprinted in Ezra Solomon (ed.), *The Management of Corporate Capital* (Glencoe, Ill.: The Free Press, 1959). Finally, any major lumpiness in discrete investment projects causes no trouble when our assumptions regarding the regularity and smoothness of the envelope of the efficient set are satisfied, and perfect capital markets are assumed throughout. Cf. Jack Hirschleifer, "On the Theory of Optimal Investment Decision," *Journal of Political Economy*, LXVI (Aug. 1958), and Martin J. Bailey, "Formal Criteria for Investment Decisions," *Journal of Political Economy*, LXVII (Oct. 1959).

2. *Op. cit.*

Corresponding to the same definition of marginal return on a capital budget, we also define the *average* (expected) *rate of return* on any given sized budget as being the *maximum* rate of discount which can be used in computing the (expected) present value of the differences in the company's cash flows attributable to the presence or absence of the entire current capital budget (treated as if it were a single project), subject to satisfying the standard present value criterion. We now *assume* a steady state in which profit opportunities are such that both average and marginal (expected) returns, expressed as a function of the size of capital budget, are constant over time. In this context, apart from transients which are irrelevant for present purposes, any average (expected) rate of return on the capital budget [3] implies that the company's net capital stock and earnings will be growing at the same (expected) rate *if* all induced cash flows are reinvested. Any smaller reinvestment would result in less growth. For any given size of current capital budget and initial capital stock, there is consequently some *maximal* amount of funds which the company *could* expect to pay out (i.e., not reinvest) at the end of the current period, consistent with expecting to be merely "no worse off" in terms of company "earning power" and (appropriately defined) "stock of earning assets" at the end of the period.

Such a maximal amount of *pro-forma* withdrawable funds is clearly the root concepts of "net income" or earnings in all modern treatments; we make it more precise by specifying "no worse off" to mean that the expected values of the corresponding maximal *pro-forma* withdrawals over all future periods be at least equal to the current period's value, and we make the earnings concept relate specifically to the current period's assets by imposing a (*pro-forma*) constraint of no outside financing in any future period. Specifically then, *corporate earnings* or *profits* for any period are defined to be equal to the maximum cash dividend which (expectationally) *could* be paid in that period consistent with (*pro-forma*) no outside financing *and* with equally large *expected values* of the (level stream of) dividends which *could* be paid in future periods subject to the same *pro-forma* financing constraint.[4]

3. *Excluding* that part required to maintain the current stock of capital and current level of earnings; see next paragraph.

4. This definition (first advanced in John Lintner, "A New Model of the Cost of Capital: Dividends, Earnings, Leverage, Expectations and Stock Prices," mimeo.—paper delivered at St. Louis meeting, Econometrica Society, Dec. 1960) has the important virtues of (a) making a 100 per cent dividend payout imply a constant (expected) earnings stream if there is no outside financing, (b) tying the concept of corporate earnings relevant to investors (and thereby market values) directly to the cash flow functionals of the investors themselves, while at the same time (c) avoiding the circularity of

This last concept is of pivotal importance. It provides the appropriate zero point on the growth scale for the rest of our analysis: since outside financing is excluded, a 100 per cent dividend payout ratio $\Longleftrightarrow$ retentions ratio zero $\Longleftrightarrow$ no growth. In addition to defining profit operationally (in terms of expectations), this concept also determines our concepts of depreciation and net investment, and establishes the interrelationships [5] between these concepts and the average and marginal rates of return introduced above which are also needed in the rest of our analysis. Specifically, *depreciation* is the amount of *gross investment required to maintain the earnings on existing assets* as defined above — this part of total cash flow ("quasi-rents") attributable to the capital stock is *not* "available for dividends" even in the "no growth" case. *Net* investment is then actual gross investment net of depreciation so defined, and henceforth "size of capital budget" will be measured in terms of net investment.

There is one further complication to be handled before we grind out some results. The position of a firm in its industry and the profitability of further new investment at any point in time depend heavily on whether it has led or lagged in research and development, in the introduction of new products, new capacity, new cost-reducing technologies, long-range advertising and other promotion of product-market position, and so on in the recent and more remote past. This is especially true in the major oligopolistic industries which account for major fractions of all plant and equipment expenditures, research and development and promotional outlays, and equity values: but it is also true quite generally, both in the short run and cumulatively in the longer run.[6] In order to encompass the essence of the problems involved in decisions for continuing growth and to incorporate essential determinants of the profit opportunities available to potentially growing firms at given points in time, the profit-possibility function (plotted against *dollar* amounts of current net investment) must depend explicitly on investments in other time

making *corporate* earnings a function of market price change (as do the standard Hicks-Alexander variants) — when market price is the thing to be explained, and finally (d) legitimately simplifying dynamic growth models through the use of concepts of *net* earnings and investment.

5. I.e., it defines the content of the penultimate footnote and hence of the associated sentence in the text.

6. See Lintner, "Effect of Corporation Taxation on Real Investment," *op. cit.*; James S. Duesenberry, *Business Cycles and Economic Growth* (New York: McGraw-Hill, 1958) and Meyer and Kuh, *op. cit.* The importance of including outlays for advertising and other promotion of product-market position in the capital budget, when the outlays are intended to affect receipts in subsequent periods, has been emphasized by Dean, *op. cit.*

periods (or as their surrogate, recent realized levels of earnings). To bring out the central implications of such dependence most simply, I shall assume [7] that the *average* profit rate p on the net investment flow (capital budget) F_t^* at any time t is constant over time when written in the form

$$(1) \qquad p_t = p(F_t^*/Y_t^*, \ . \ . \ . \ .) = p(f, \ . \ . \ .) = p \text{ constant over time,}$$

where Y_t^* is the corporation's aggregate net earnings at time t, and $f = F_t^*/Y_t^* =$ constant. (For simplicity, the function is stated here in the form appropriate for conditions of certainty; the appropriate modifications for stochastic conditions are introduced in Sections III and IV.)

To avoid confusion and misunderstanding later, however, it is necessary to examine closely the sources of the profitability included in "p" in this function. If we had assumed no time-interdependence in the profitability function, the incremental cash flows due to a net capital budget of a given dollar size would be only those directly attributable to the included investments in the usual manner. But with equation (1), there is a *further* increment of profitability attributable to a given sized net investment in a given current period: the profit-possibility function plotted against *dollars* of investment is shifted to the right, with the result that the number of dollars of (gross) investment henceforth required merely to maintain any given sized stock of capital intact is reduced, and this reduction in future outflows is also part of the increment in corporate earnings due to the current investment. The "p" in equation (1) thus includes *both* the direct *and* this induced profitability for each size of net investment in the given current period.

The reader should also note carefully that the "induced" component of profit on any given amount of net investment in any single current period reflects a *level stream* of no-longer-required outlays in all future periods. The induced components of profitability in function (1) will grow over time *if and only if* net investment is positive in future periods (so that capital stock and earnings also grow), but *this* growth, if present, would reflect *future periods'* investments — or more generally a continuing investment *policy* —

7. This function has been used in the earlier work of Gabriel A. Preinreich, *The Nature of Dividends* (New York: Columbia University, 1951); J. B. Williams, *The Theory of Investment Value* (Cambridge: Harvard University Press, 1938); Myron J. Gordon and Eli Shapiro, "Capital Equipment Analysis: The Required Rate of Profit," *Management Science*, III (Oct. 1956), reprinted in Ezra Solomon (ed.), *The Management of Corporate Capital*, *op. cit.*; and Myron J. Gordon, *The Investment, Financing, and Valuation of the Corporation* (Homewood, Ill.: Irwin, 1962).

and would not be a proper component of the profitability of any given *current* period's investment, and hence is not reflected in "p".

The other properties of equation (1) may now be stated. Since the average profit rate p will not in general be invariant with respect to the relative size of the current investment f, we have $p'(f) \leqslant 0$ but because equation (1) is constant over time as a function of f, this derivative will be also. Finally, we define the marginal rate of return in the usual way [8] as the partial derivative of the total dollar return with respect to the quantity of funds invested, which gives (with Y^*_t fixed)

$$\text{(1a)} \qquad \rho_t = \frac{\partial F^*_t \; p(F^*_t/Y^*_t)}{\partial F^*_t} \equiv \frac{\partial f p(f)}{\partial f} = p(f) + f\,p'(f).$$

The marginal rate of return will also be a constant function of f over time with the further property that $\dfrac{\partial \rho}{\partial f} \leqslant 0$.

Since in this paper we are considering only internally financed growth, all debt, leverage and new equity issues are appropriately excluded, and henceforth we have $f \equiv r$.

II. The Cost of Capital and Optimal Dividends and Growth Under Certainty [9]

With no outside financing, only a single decision determines the value of all three variables x, r and f. Since we seek the optimal target or steady state value of these variables, we shall *assume* that, once determined, the value of the decision variable r (or x, or f) will be *held constant* over all future time. But for decision-making purposes, r is a true variable whose value is to be chosen. The *criterion* ordering the desirability of the outcomes of alternative choices of x or r is the market prices of the common equity, which are a function of the alternative streams of cash flows to investors, which in turn depends on the profit function (1) and the decision variable r. Analogous to the traditional theory of production, we will now derive decision rules in terms of (in)equalities between marginal rates of

8. This perhaps needs emphasis, since as shown (three footnotes below), ρ is also equal to the marginal growth rate, which might otherwise seem something like pulling a rabbit out of a hat.

9. Some of the last few pages and this section overlap with Section II of Lintner, "The Cost of Capital and Optimal Financing of Corporate Growth," *op. cit.* The common material was needed there in summary form to develop implications for other forms of financing; here I develop the basic model more rigorously and examine its "cost of capital" curves, decision rules and growth implications in substantially greater depth and detail.

return and marginal costs of capital as defined above, using the constancy assumptions previously made, and assuming there is no uncertainty.

Note first that using continuous compounding for convenience, so that Y_t^* is the instantaneous *rate* of earnings flow, the rate of growth g of Y_t^* is

(2) $g = g_{Y_t} = d \log Y_t/dt = r\,p(r)$.[1]

Moreover, since aggregate dividends are $D_t^* = xY_t^*$, where any chosen $x = 1 - r$ has been assumed to be constant over time, it is clear that the growth rates of dividends and earnings will be equal and constant. The aggregate dividend distribution at any time t will be

(3) $D_t^* = xY_t^* = xY_0^* e^{gt}$.

Even though it is patently unrealistic to assume steady growth forever along with certainty (just as it is to assume away taxes and other financial alternatives and costs), the very simplicity of the setup highlights certain basic relationships of quite general significance, as we shall see.

Turning now to our stock price criterion function for the choice of best retention ratio, investment and growth rate, we adduce well-known theorems for equilibrium in perfect markets which require that the sum of current cash returns (here dividend yields $= D_t/P_t = y_d$) plus rates of growth in own price for all assets equal the current market rate of discount, i.e., $y_{dt} + d \log p_t/dt = k_t$. The solution of this differential equation for market price, with k constant, is

$$(4) \qquad P_t = \frac{D_t}{k-g} = \frac{D_0 e^{gt}}{k-g} = \int_t^{\infty} D_t e^{-(\tau - t)(k-g)}\, d\tau,\ k > g.$$

By derivation, the prices given by this equation will satisfy the basic equilibrium theorem cross-sectionally over different stocks and securities, *and* will do so continuously over time.[2] For the *current* price of the stock, P_0, equation (4) reduces to

1. This step may be justified by passing to the limit in the discrete case (underscoring denoting flows or rates measured in discrete periods) where, under present assumptions, $\underline{Y}_{t+1}^* = \underline{Y}_t^* + r\,p(r)\underline{Y}_t^*$ or $\Delta Y_t^*/\underline{Y}_t^* = r\,p(r)$ which gives (2) in the limit. Alternatively, (2) is implied by $\underline{Y}_{(t+\tau)}^* = \underline{Y}_t^* + r\underline{Y}_t^*\,\tau\underline{p}(r) + 0(\tau)$ as $\tau \to 0$.

2. It is apparent that the condition $k > g$ is both *necessary and sufficient* to insure the convergence of the integral of the primary differential equation. See [A1] for proof that the properties stated hold regardless of the varying time patterns of *any* of the variables (so long as a corresponding condition on the time of integral k_τ and g_τ is satisfied).

(4a) $$P_0 = \frac{D_0}{k-g} = \frac{xY_0}{k-g}, \quad k > g,$$

and differentiation with respect to r gives (recalling equation (2) and that $r = 1 - x$)

(5) $$\frac{dP_0}{dr} = P_0\left[-\frac{1}{x} + \frac{dg/dr}{k-g}\right] \gtrless 0$$

$$\text{as } \frac{dg}{dr} \gtrless \frac{k-g}{x} = Y_0/P_0 \equiv y_e,$$

where y_e is the current earnings yield of the stock. But since $g = g(r) = rp(r)$, the marginal growth rate dg/dr is equal to the *marginal internal rate of return* $\rho \equiv \rho(r)$.[3] Equation (5) consequently can be written

(5a) $$dP/dr \gtrless 0 \Longleftrightarrow \rho(r) \gtrless y_e. \Longleftrightarrow x\rho(r) \gtrless y_d.$$

Note that maximizing stock price in this growth model requires management to continue internal investment *beyond* the point where the *"average return"* or *profit rate* has been maximized, just as in the standard cases in the literature.[4] Our results, however, go considerably beyond this orthodox property by showing that adherence to the criterion of maximizing shareholders' equity, in the context of separation of ownership and management as emphasized by Berle and Means, does *not imply maximizing growth*[5] — as has generally been proposed elsewhere — since the latter criterion would require continuing investments until dg/dr had been reduced to zero instead of only to $y_e > 0$. We may anticipate later results to add that this conclusion also is completely general — and indeed holds by even wider margins once important "realistic" complications are added.[6]

3. Alternatively, by definition

$$p(r) = r^{-1}\int_0^r \rho(r)dr \text{ or } g(r) = rp(r) = \int_0^r \rho(r)dr$$

and the identification in the text follows by direct differentiation.

4. See Lutz, *op. cit.* (esp. Chap. II). Maximizing profit rates requires $p' = 0$ which by (1a) implies $\rho_0 = p$ and the standard model to maximize capital value requires $\rho_0 = k \leqslant p$. It should be noted that our present model (sustained growth with certainty) requires *still further* investment to $\rho = y_e < k < p$. However, on this latter point see sections below.

5. After this paper was drafted, the same observation was made by William Baumol, "On the Theory of Expansion of the Firm," *American Economic Review*, LII (Dec. 1962).

6. Incidentally we should note that the common complaint that it just doesn't make sense to tell I.B.M., say, with its recent 18 per cent per annum growth rate, to push its capital budgets to the point where the return on *marginal* investments is as low as the earnings-yield on its stock (only recently under 1.5 per cent), *because* "you can't maintain your growth rate that way" completely misses the mark. The conclusion that marginal returns should

We next consider the shape of the "cost of capital" function for unlevered firms under certainty. Since the cost of capital to be compared with ρ in this case is y_e, we need merely examine its shape plotted against the retention ratio r. From $y_e = (k - g)/x = [k - rp(r)]/x$ it is clear that at $x = 1$, (or $r = g = 0$) we have $y_e = k$. Moreover,

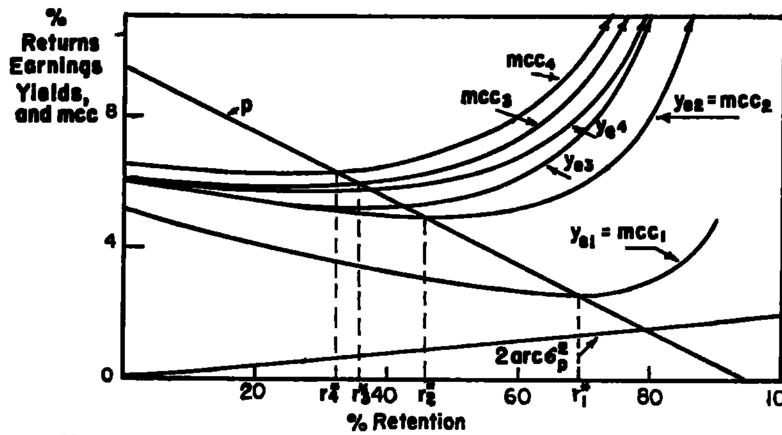
$$\text{(6)} \qquad \frac{\partial y_e}{\partial r} = \frac{x(-\rho) + (k - g)}{x^2} = \frac{y_e - \rho}{x} \lesseqgtr 0 \Longleftrightarrow \rho \gtreqless y_e$$

and

$$\text{(6a)} \qquad \frac{\partial^2 y_e}{\partial r^2} = \frac{2(y_e - \rho) - x\,\partial\rho/\partial r}{x^2} = \left[2\frac{\partial y_e}{\partial r} - \frac{\partial \rho}{\partial r}\right]/x.$$

FIGURE I

ILLUSTRATIVE MARGINAL (EXPECTED) RATES OF RETURN, EARNINGS YIELDS, AND MARGINAL COSTS OF CAPITAL: COMPARISONS OF MODELS I–IV.



Note: Data from Table I.

Consequently, when $r > 0$, y_e *falls* with increasing r, though at generally diminishing rates, as long as $\rho > y_e$ and thereafter rises at increasing rates when the marginal return $\rho < y_e$.[7] With growth steady, certain, and eternal, the optimizing decision rule in (5a)

not be pushed so low is correct, *either* because of uncertainty *or* expectations that relative investment opportunities will deteriorate in the future; *in the absence of these factors* the reason usually given is actually the reverse of the truth as may be seen by noting that, if internal funds were sufficient, any company would *increase* its growth rate to a maximum if it pushed investments to a marginal return of zero! This proposition depends only on the first equation in (1a) and equation (2); it does not depend on the particular form of profit function used in this paper.

7. The condition $\rho_0 = y_e$ clearly makes the corresponding value of y_e a true minimum since this makes $\partial y_e/\partial r = 0$ and $-\partial\rho/\partial r > 0$ because of the diminishing marginal returns assumed, so that $\partial^2 y_e/\partial r^2 > 0$ at this point. It

ILLUSTRATIVE TABLE I

ILLUSTRATIVE MARGINAL (EXPECTED) RATES OF RETURN, EARNINGS YIELDS, AND MARGINAL COSTS OF CAPITAL: COMPARISONS OF MODELS I–IV.

		k or $\omega = .05$ $\sigma'_{ij} = .005$			$p = .095 - .05r$ $C\sigma^2_{p_0} = .005$		$a = 2$ $\beta = 2$			
r	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
p	.095	.090	.085	.080	.075	.070	.065	.060	.055	.050
g	0	.009	.017	.024	.030	.035	.039	.042	.044	.045
ρ	.095	.085	.075	.065	.055	.045	.035	.025	.015	.005
$y_{e1} = mcc_1$	.05000	.04556	.04125	.03714	.03333	.03000	.02750	.02666	.03000	.05000
$y_{e2} = mcc_2$	.06000	.05666	.05375	.05142	.05000	.05000	.05250	.06000	.08000	.15000
y_{e3}	.06000	.05677	.05425	.05271	.05266	.05500	.06150	.07633	.11200	.23100
$2arC\sigma^2_{p_0}$	0	.00200	.00400	.00600	.00800	.01000	.01200	.01400	.01600	.01800
mcc_3	.06000	.05877	.05825	.05871	.06066	.06500	.07350	.09033	.12800	.24900
y_{e4}	.06311	.06057	.05869	.05786	.05849	.06138	.06802	.08243	.11537	.2226
λ_4	1.0373	1.0575	1.0742	1.0953	1.1219	1.1463	1.1797	1.2037	1.2251	1.259
mcc_4	.06546	.06405	.06304	.06337	.06562	.07036	.08024	.09922	.14134	.2803
T_1 (years)	20.00	24.39	30.30	38.46	50.00	66.66	90.90	125.0	166.7	200.0
T_2 "	16.67	19.60	23.25	27.78	33.33	40.00	47.61	55.55	62.50	66.66
T_3 "	16.67	19.56	23.04	27.10	31.64	36.36	40.65	43.66	44.64	43.29
T_4 "	15.272	17.34	19.81	22.53	25.40	28.42	31.15	33.56	35.37	36.27

consequently ***minimizes the earnings yield*** on the stock — the cost of capital in this case — and maximizes the price-earnings ratio. These relationships are all illustrated in Figure I. The y_{eI} curve continues to fall until it is intersected from above by the marginal rate of return ρ. (The other y_e curves relate to models which will be introduced later.)

All this looks very similar to the standard (static) theory of production. But another perhaps less expected property of our model should also be noted. This model, and in particular equation (5a), says that *if* a corporation were operating under conditions of certainty and no taxes, and had investment opportunities permitting *constant* rates of growth into the unlimited future, it should make investments up to the point where the marginal internal rate of return on current investment no longer exceeds the current earnings yield of the stock — and this rule is valid *even though* in the usual case when the company is paying some dividends and marginal returns ρ are less than average returns p, it is also true [8] that the current earnings yield y_e is necessarily *less* than the discount rate k which reflects the returns available on the shareholder's alternative investments. (Note that this result does not depend on any tax differentials!) The apparent paradox is resolved by noting that equation (5) also can be rewritten as

$$\text{(5b)} \qquad dP/dr \geqslant 0 \Longleftrightarrow x\rho(r) + g(r) \geqslant k,$$

i.e., the marginal return for the *investor* from added investment within the company is equal to the sum of the dividend payout applied to the marginal internal return within the company on current investments *plus* the growth rate on the retention itself, and if this *sum* is greater than the discount rate, he will prefer the retention. The fundamental significance of this result is the emphasis it gives to the distinction between marginal rate of return requirements on current investments at every point in time in growth situations *to a company* and marginal returns *to investors*.

Alternatively, one could of course, if he wished, explain the "paradox" of our $\rho < k$ by noting that changes in the decision

is also clear that y_e rises at necessarily increasing rates beyond this point since $\partial y_e/\partial r$ is then also positive in (6a). It is also clear that $\partial^2 y_e/\partial r^2$ must generally be positive as well for $0 \leqslant r < r^*$ (where r^* is the optimizing value corresponding to $\rho = y_e$) — i.e., y_e must decline at generally diminishing rates in the region as asserted in the text — because it does reach a minimum at $\rho = y_e > 0$ and ρ is continuously declining. But if ρ is sufficiently large relative to y_e at low retention rates it is possible for y_e to be declining at increasing rates in a relatively narrow interval in this region.

8. Since $y_e = (k - g)/x$ can be written $xy_e + (1 - x)p = k$, we have with $0 < x < 1$, $p \geqslant k \Longleftrightarrow y_e \leqslant k$; but (5b) can be written $k \leqslant p - (p - \rho)x$, so $\rho \leqslant p \Longleftrightarrow p \geqslant k$ with $x > 0$.

variable x or r assumed constant over time, involve changes not only in the current investment rate but in all future investment rates, and then define a *different* marginal rate of return to the company on the change in r as the partial derivative of the integral of all changes in dollar earnings from *all* changes in amounts of investments *throughout the future* with respect to the integral of all increments of dollar investments throughout the future — this instead of our ρ (cf. the first part of equation 1a). The cutoff rule for *this* marginal return would be the standard discount rate k even in growth situations, and the simple "discount rate rule" for internal company use would be saved — but only at the expense of major complications in the calculation of the marginal return to be compared with k. While these latter complications in the mathematics can be handled without too much difficulty in the strict certainty case, they compound rapidly as successively more realistic types of uncertainty are considered (since the parameters of the stochastic processes of growth rates *and* future investment opportunities (a) differ from the parameters of uncertainty on current profit rates *and* (b) vary over time *and* (c) in anticipation must be adjusted for (investors', not company) utility considerations either directly or through derivative market-equivalence functions).

The decision rules advanced above in terms of our ρ[9] are not only strictly rigorous but can be adapted to the more realistic uncertainty cases (our primary objective) much more straightforwardly precisely because these different elements responding differently are kept distinct. Also, companies need decision rules which can be applied to the (expected values and variances of) current judgments on the profitability of current investments even in dynamic growth situations — and our ρ (as developed in subsequent sections) provides these. As we develop our models explicitly recognizing uncertainty from this form of the certainty model, we require judgments of a type company managements can perhaps be expected to make — such as judgments that the expected value and variance of profit rates on current budgets are about so much, that future uncertainty can be expected to evolve roughly in some simple way, but that *expected* profitability in the future relative to then-attained company size and size of budget will be about the same as

9. It should perhaps be re-emphasized that our ρ fully accounts for the shift in the investment opportunity function *throughout the future* which arises from the *current* net investment — it differs from the alternative only in that it does not incorporate the returns attributable to all the *further* shifts due to all the *further investments* to be made in all *future* periods. Cf. p. 57 above.

currently, that the market-place reflects about so much risk-aversion by investors, etc. — and not judgments of hopelessly complex composites directly.

We now proceed to our essential objective of developing models which explicitly recognize and incorporate the omnipresent real world fact of uncertainty. But if it needs to be said again, I will do so: this *latter* task is the *sole* purpose of the preceding models under certainty — the assumptions of prescience and possibility of undiminishing rates of growth until, through and beyond the hereafter (as well as omissions of such mundane matters as taxes!) are so completely extreme that the decision rules are *not* intended for practical use by businessmen in the form given.

III. Simple Stochastic Unlevered Growth

I have elsewhere[1] emphasized that, once uncertainty is admitted, the problem of determining the best capital budget of *any* given size is formally identical to the solution of a security portfolio analysis, and shown[2] that the "shadow value" of any expected return is equal to the row-sum of the inverse of the variance-covariance matrix with all other assets in the portfolio (in the present application, all elements of the existing capital stock and other projects included in the capital budget). Consequently, although the same marginal return requirements provide a valid decision rule for both the composition and size of capital budgets under certainty, this is no longer true under uncertainty since the required marginal expected return will vary from project to project (even relative to their own σ and σ^2). In the rest of this paper, I shall focus on the optimal determination of the *size* of the capital budget, simply assuming that a Markowitz-type "efficient set" analysis has already been made which yields a three-dimensional (per time period) "profit-possibility function" relating amount of investment (size of budget), expected average profit rates and variance of return.

The "Profit-Possibilty" Function. For each possible size of the investment budget, this profit-possibility function is the *envelope* of all individual investment possibilities consistent with the budget-size constrants; for each possible size of budget it gives the *maximum expected* return obtainable at each level of variance of re-

1. "A New Model of the Cost of Capital," *op. cit.*, and "The Cost of Capital and Optimal Financing of Corporate Growth," *op. cit.*

2. "The Valuation of Risk Assets and the Selection of Risky Investments," *Review of Economics and Statistics*, forthcoming.

turn, and the *minimum* return variance obtainable at each level of expected return on new invested funds.

The position of this envelope depends on the size and composition of the stock assets already held which are reflected in current sustainable net earnings before interest. In addition, however, there is evidence that if profits have risen above or fallen below expected values, the relative amounts of current investments which can be made with a given expected return will also be respectively enlarged or depressed. With contemporaneous observed aggregate earnings $Y_t^* = \hat{Y}_t^* + \epsilon_{Y_t}$, the general equation for the profitability function is $\psi_{0_t}[F_t^*/Y_t^*, \hat{p}_t, \hat{\sigma}_{p_t}^2] = 0$ where $\hat{p}_t$ is the expected average return per dollar invested, and $\sigma_{p_t}^2$ is the expected variance of the rate of return on this investment, $\hat{}$ indicating expected values in each instance. In keeping with my continuing emphasis on (expectationally) steady growth, however, I assume specifically that $\psi_1(r, \hat{p}, \hat{\sigma}_p^2) = 0$ is invariant over time with $r = F_t^*/Y_t^*$ constant at some level to be determined. As in the certainty case, I assume diminishing (average *and* marginal) expected profitability with increasing budget size at any point in time (i.e., $\partial\hat{p}/\partial r \leqslant 0$ and $\partial\rho_{\hat{}}/\partial r \leqslant 0$), which are both constant over time for any r and σ_p^2, as is the marginal expected rate of return $\rho_{\hat{}} = \partial r\hat{p}/\partial r$. The envelope function ψ_1 has the further properties $\partial\sigma_p^2/\partial r \geqslant 0$ and $\partial\hat{p}/\partial\sigma_p^2 \geqslant 0$, both also constant over time for fixed values of the other terms in ψ_1, and finally I assume that the usual "concavity" conditions involving second derivatives are satisfied. In order to keep the analysis within manageable limits for present purposes, however — and to focus more sharply on the returns required in the presence of given risks and simplify the development — I will work throughout this paper with the two-dimensional contour of this profit-possibility function traced out by the intercepts of a plane of constant (prespecified) variance.

The Simple Stochastic Process. Assuming that management adjusts the mix of its capital budgets to keep $\hat{\sigma}_{p_t}^2 = \hat{\sigma}_p^2$ constant over time,[3] our stochastic profit function in explicit form is now

$$\hat{p}_t = \hat{p}_t(r_t, \hat{\sigma}_{p_t}^2) = \hat{p}(r) \text{ a constant function} \tag{7}$$

over time for any given $\hat{\sigma}_p$; and we assume specifically that actual

3. Our certainty model — Model I — was a limiting case with $\hat{\sigma}_p^2$ set equal to zero.

rates of profit on new investments at any time t, $p_t = \hat{p}_t + \epsilon_{p_t}$, are normally distributed with (constant) variance σ_p; and that ϵ_{p_t} are *independent* over time.[4] Since $g_t = r\,p_t(r)$, $\sigma^2_{gt} = r^2\sigma^2_p$, but we must also allow for the fact that even with no growth (net investment $r = 0$) there is variance σ_{pt_0} in the profit rate (and hence fluctuations in $\tilde{g}_t$ about its mean zero). These considerations suggest $\sigma^2_{gt} = (1 + r^2_p)\,\sigma^2_p$, but for greater generality we shall use the form $\sigma^2_{gt} = (1 + ar^2_p)\sigma^2$ with $0 \leqslant a$ to allow for the fact that the uncertainty of growth rates may not increase in more or less full proportion to increased retentions due to covariance between profits on newly added and prior assets. Under these assumptions: (1) the expected *average* rate or profit over a period t units in length $E[p_t]$ will be equal to the expectation at each point $\hat{p}$, and (2) the same holds for the growth rate $E[\bar{g}_t] = \hat{g}$ (since $g_t = rp_t$ and r is constant); (3) the variance of the average profit and growth rates vary inversely with the period of the average, since shocks in both rates are time-wise independent by assumption, so that, in particular, var $(\bar{g}_t) = (1 + ar^2)\,\sigma^2_p/t = \sigma^2_{gY}/t$; but (4) and this is most important, the variance of the cumulated or total growth over a period t *increases linearly* over time, since var $(t\,\bar{g}_t) = t^2$ var $\bar{g}_t = t(1 + ar^2)\,\sigma^2_p = t\,\sigma^2_g$.

Our problem now is to determine the set of values of $\hat{p}$ (or ρ_{Λ}) and r which will maximize $\hat{P}_0$, the expected value of current share price, subject to (7) and a fixed (preselected) $\hat{\sigma}^2_p$, where $P_t = \hat{P}_t + \epsilon_{P_t}$ with ϵ_{P_t} distributed with mean zero and independently over time.[5] The optimal $\hat{p}$ and r are simultaneously determined by finding those values which maximize $\hat{P}_0$. Since the crucial decision is at what point to stop expanding the capital budget which is determined by the point where marginal returns are no longer sufficient to justify expansion, we shall in particular seek a decision rule for this part of the maximizing problem which is stated in terms

4. The p_t thus represent repeated drawings from a stationary (normal) distribution with constant variance σ^2_p per unit time period. Again, this assumption would be quite inappropriate if we were focusing on short-run cyclical phenomena, but it provides a convenient and simple benchmark assumption in our longer-run context. A more general model is examined in the following section.

5. This assumption would clearly be inappropriate for a study focused upon short-term stock market price determination, but it provides a reasonable basis for an analysis focused upon optimal long-term company behavior. I am *not* undertaking to analyze how management could best accentuate and capitalize upon favorable short-run speculative swings in stock prices.

of *marginal cost of capital* — which, it will be recalled, is *defined* as the *minimum marginal expected rate of return on current investment required* to justify *both* the *acquisition and investment* of any further capital. Since this rule on our criterion must be derived from the conditions for maximizing the current value of the stock, I now turn to the matter of stock prices under conditions of uncertainty.

The Stock-Price Model. It is immediately obvious that the theorems respecting relative market values in classic competitive equilibrium all remain valid under uncertainty so long as it may be assumed that investors base their purchase and sale decisions upon present values (at risk-free discount rates) of the certainty equivalents of the elements of uncertain income streams. (The *certainty equivalent* of a random receipt is *defined* to be that single value which, if certain, would be equivalent in the decision-makers mind to the uncertain prospect represented by the full distribution of the random element.) It seems entirely reasonable to believe that this model appropriately summarizes much of the essence of the behavior of risk-averse investors — and the prevalence of diversification establishes the predominance of risk-aversion.[6] I have elsewhere[7] shown that — when every individual investor in the market holds that combination of cash, savings deposits with riskless positive return k, and risky securities which *he* most prefers, given his wealth, utility function and (multivariate normal) probability judgments over random outcomes; when these probability judgments are the same among investors;[8] and when the prevailing market prices of all risk assets are established in purely competitive markets which, as Arrow has shown,[9] yield a Pareto-optimal allocation of risks — this "present value of certainty-equivalents" model is rigorously valid with respect to the relative values of securities in the optimizing portfolios of risk-averse investors over any single holding period of arbitrary length.[1] Specifically, where

6. Indeed, Arrow has shown that risk aversion is a necessary condition for competitive equilibrium in markets for risk assets. Kenneth J. Arrow, *The Role of Securities in the Optimal Allocation of Risk Bearing* (Cowles Commission Papers, New Series, No. 77), 1953.

7. "The Valuation of Risk Assets and the Selection of Risky Investments," *op. cit.*

8. Note that wealth and utility functions need not be the same. The assumption of common probability judgments may or may not be a necessary condition; in any event, it establishes the theorem and brings out the implications of uncertainty per se, as distinct from diverse judgments.

9. *Op. cit.*

1. The theorem derives relative aggregate market values over a fixed set of available issues; but for any given (fixed) number of shares of each outstanding, this is sufficient to determine relative market *prices.*

the security is assumed sold at the end of any period, the present price is the risk-free present value of the certainty equivalent of the sum of the cash (dividend) receipt during the period and end-of-period sales price. The latter, by an extension of the same line of analysis, clearly may be regarded as the then-present-value at the riskless rate of the certainty equivalents of dividends in the second period and resale value at the end of the period. By iteration, one gets current price as the present value of the certainty-equivalents of the dividend stream itself over time — *except* that the relevant risk parameters are greater than those of the dividend stream alone for two reasons. Allowance must be made for the greater (but functionally related) variance of stock prices around their expected values at all future times. Moreover, within any period there is foreknowledge that new information will become avaliable.[2] This information may be either favorable or unfavorable, and its effect on prospective resale values, to a first approximation at least, can also be expected to be roughly proportional to the uncertainty of the underlying income stream. For both reasons, the *relevant* uncertainty is some multiple $c > 1$ of the uncertainty of the underlying dividend stream of the security.

For purposes of the present paper, I am simply assuming that investors will behave in terms of a corresponding "certainty-equivalent" model with respect to the allocation of funds over the risk assets in their portfolios — and that the relative market prices on different securities will be determined by the certainty equivalents of the probability distributions of their prospective yields to investors — when investors are assumed all to have hyperbolic utility functions and to form their probability judgments in terms of the simple stochastic process[3] given above. The latter assumption, of course, makes dividends themselves, future stock prices, *and* rates of return (dividend yield *plus* relative gain or loss) lognormal. The hyperbolic form of utility function, used earlier in somewhat related work by Tinbergen,[4] has many desirable properties and avoids

Moreover, the relation between "expected value" and "certainty equivalent" on a given stock can perfectly well reflect the covariances with other stocks; "certainty-equivalents" models do not necessarily ignore intercorrelations as frequently preseumed. In what follows we shall recognize nonzero covariances but for simplicity assume them positive and fixed in value.

2. This is, of course, distinct from any time dependence among the stochastic elements of the underlying stream. As developed in Section IV below, it is quite possible within the framework of a certainty-equivalence model to allow directly for much of the impact of this latter factor.

3. The same assumption will be made with respect to the more complicated stochastic process introduced in Section IV.

4. Jan Tinbergen, "The Optimum Rate of Saving," *Economic Journal*, LXVI (Dec. 1956).

the nonsense implications of the quadratic and exponential functions traditionally relied upon to represent risk-averse behavior. Apart from irrelevant choices of origin and scale, the general form of the hyperbolic is $U\ (\tilde{.}) = -\ (\tilde{.})^{-a}$ where $a > 0$ implies an upper bound (as I shall assume) and measures the degree of risk-aversion.[5]

Then with $\tilde{D}_t = \tilde{D}_0 e^{\tilde{g}t}$ we have [6]

$$(8)\qquad E[U(\tilde{D}_t)] = E(\tilde{D}_t^{-a}) = -D_0^{-a} e^{-at(\hat{g} - \frac{a\sigma^2_g}{2})}$$

so that the certainty equivalent

$$(9)\qquad \check{D} = D_0 e^{t(\hat{g} - \frac{a\sigma^2_g}{2})}.$$

Also for later reference, note that if we were to treat each stock in isolation, assume that the variance of the *expectations* of the *sum* of dividend receipt and absolute ΔP_t in period t is a multiple $c' > 1$ of the variance in D_t, the price of the stock would be

$$(10)\qquad P_0 = \int_0^\infty D_0 e^{-t[k - \hat{g} + \frac{ac'_g}{2}\sigma^2]} = \frac{D_0}{k - \hat{g} + \frac{a'}{2}\sigma^2_g},$$

$$k + \frac{aC'}{2}\sigma^2_g > \hat{g}.$$

Now note that if any investor were choosing the best portfolio in which to place a given amount of funds ΣV_0 for long-term investment on the basis of his hyperbolic utility function (which has the property of constant *proportional* risk aversion [7]), and the portfolio was to be chosen from an (infinite) set of possible portfolios, each of whose yields were lognormally distributed, he would choose the one which would provide the greatest certainty equivalent of yield. This would be that portfolio for which $\theta = \bar{g}_p - a\sigma^2_{g_p}/2$ is as great as possible, where $\bar{g}_p$ is the expected average rate of growth in value of the portfolio and $\sigma^2_{g_p}$ is its variance. Moreover, if the component stocks have multivariate lognormal yield distributions,[8] then it

5. Cf. John W. Pratt, "Utility and Aversion to Risk," mimeo., Harvard Graduate School of Business Administration, 1962.

6. J. Aitchison and J. A. C. Brown, *The Lognormal Distribution* (Cambridge, England: Cambridge University Press, 1957), p. 8.

7. Cf. Pratt, *op. cit.*

8. It is quite true that linear mixtures of lognormally distributed variates are not themselves strictly lognormally distributed. I have simply assumed that investors treat them as being so. Several justifications may be offered. The prices of large diversified industrial and utility stocks (which are notorious "poolers of risks") appear to be as lognormal as the "common generality" of companies, as do the changes in values of some 125 investment

can be shown[9] that if f_i represents the proportion of total ΣV_0 allocated to the i^{th} stock and $\Sigma f_i = 1$ we can identify $\bar{g}_p$ with $\Sigma_i f_i \bar{g}_i$ and $\sigma^2_{g_p}$ with $[\Sigma_i f_i^2 \sigma^2_{g_i} + 2\Sigma_i \sum_{j>i} f_i f_j \sigma_{ij} + \Sigma_i f_i \sigma^2_{g_i}]$. The best combination of stocks maximizes the Lagrangian function

$$\Sigma_i f_i \bar{g}_i - (a/2)\, [\Sigma_i f_i^2 \sigma^2_{gi} + 2\Sigma_i \Sigma\, f_i f_j \sigma_{ij} + \Sigma_i f_i \sigma^2_{gi}] - \omega[\Sigma_i f_i - 1]$$

which yields the typical equation

$$(11) \qquad (1 + f_i) a\sigma^2_{g_i} + a\sum_{j \neq i} f_j \sigma_{ij} = \bar{g}_i - \omega : i = 1 \;.\;.\;.\; m \text{ stocks}$$

to determine the relative holdings f_i.

But changes in any company's investment policy and retentions rates change the expected value and variance of its growth rate. Particular interest consequently attaches to the different *combinations* of g_i and $\sigma^2_{g_i}$ which will at least maintain the f_i and hence the relative value of its stock. Assume now that all investors hold identical multivariate lognormal distributions over the $\tilde{g}$'s, and that all have the same risk-aversion coefficient a; then each investor will hold the same mix of stocks in equilibrium (although the actual total investment in the mix will vary from one investor to another), and the f_i in equation (11) can be interpreted as the ratio of the aggregate market value of the i^{th} stock to the aggregate market value of the total portfolio of all investors (and all f_i's will be positive). If now we focus our attention on the case where these retention and investment decisions do not affect the *covariances* between the i^{th} stock and all others, which is a quite reasonable one,[1] and the effects of changes in σ^2_i and $\bar{g}_i$ on the holdings of this stock f_i and of all *other* stocks, f_j is negligible, we find that there is a *linear*[2] *indif-*

trusts which have been examined over one, five and ten year periods. Since finite mixtures of lognormal variables are neither strictly lognormal *nor* Gaussian (the central limit theorem, after all, applies strictly *only* in the limit), a choice here (as in all other statistical work) must be made and the lognormal is a very reasonable choice of a simple form. Finally, our results do *not* depend on *normality* in the logs, but are very much more general, though detailed proofs must be reserved for another occasion.

9. The step involves writing $\widetilde{e^{g_p}} = \Sigma_i f_i e^{g_i}$, squaring both sides, using a series expansion on the exponentials and $\Sigma_i f_i = 1$, taking expectations and equating like terms. See [A2] in mathematical appendix.

1. If, for instance, one assumes that random realized growth rates $\tilde{g}_i = a_i + b_i \tilde{u} + \epsilon_i \tilde{z}_i$ where a_i, b_i and ϵ_i are given constants, $b_i > 0$, $\epsilon_i > 0$ and the random variables $\tilde{u}$ and $\tilde{z}_i$ are mutually independent with mean 0 and variance 1, the $\tilde{z}_i$ shifting with policy decisions in the i^{th} firm. Then: cov. $[\tilde{g}_i - \bar{g}_i, \tilde{g}_j - \bar{g}_j] = b_i b_j$ independent of ϵ when $i \neq j$.

2. With our assumption of identity in probability distributions and utility functions, this linearity follows directly from equations (11).

ference relation between expected growth rates $\bar{g}_i$ and variances σ^2_g, which will maintain the value of the i^{th} stock in a full portfolio-optimization context. This function may be written

$$\hat{g}_i - [a(1+f_i)/2]\sigma^2_{g_i} - \underline{\sigma}_{ij} - \omega = 0 \tag{12}$$

where a is the risk aversion coefficient, σ_{ij} is a (constant) parameter reflecting the weight sum of all covariances and ω is the Lagrangian multiplier introduced earlier.

Now revert for a moment to the certainty case with $\sigma_i = 0$. The classical market equilibrium conditions require, as seen above, that $(D_t/P_t) + d \log P_t = k$ or $y_d + g = k$. With uncertainty introduced in a certainty-equivalent model, the certainty equivalent of the sum of the dividend yield and the growth rate must equal this rate. For any given market price, we have the *expected* rate of return equal to the sum of the *expected* dividend yield and expected growth rate. Since the uncertainty of the absolute dividend (and therefore the dividend yield, *a fortiori*) varies directly with the uncertainty of the growth rate, the relevant σ is some $c\sigma_g$, with $c > 1$, where σ_g is the standard deviation of the growth rate g. Therefore, we have

$$\text{C.E. of } (y_d + g) = (\hat{D}_t/P_t) + \hat{g} - C\sigma^2_g - \sigma'_{ij} - \omega. \tag{13}$$

where $C = ca(1+f_i)/2$ and the prime on σ'_{ij} reflects the "c-factors" and the summing of alien covariances. We now make the assumption that the position of this indifference curve in the $\hat{g}$ and σ plane is stable over time — which is surely a reasonable and appropriate assumption given the purposes of this analysis and the context of the rest of the model — and integration yields

$$P_0 = \int_0^\infty D_0 e^{-t[\omega - \hat{g} + \sigma'_{ij} + C\sigma^2_g]}\,dt = \frac{D_0}{\omega - \hat{g} + \sigma'_{ij} + C\sigma^2_g} \tag{14}$$

where $\omega + \sigma'_{ij} + C\sigma^2_g > \hat{g}$.

Equation (14), then, is a "present-value of the certainty equivalents of the stochastic cash-flow (dividend) stream" model of stock prices in which the certainty equivalents are determined by the equivalences along indifference curves reflecting the optimizing behavior of risk-averse investors in purely competitive frictionless markets. It should also be noted that this stock price model has essentially the same form as equation (10) derived by assuming that investors form their probability judgments in terms of the simple stochastic process given above (making the distribution of returns and dividends lognormal), and that they determine the relevant certainty-equivalence-indifference curves in terms of their own hyperbolic utility function. Equations (10) and (14) differ by the explicit inclusion of the term σ'_y reflecting covariances and the use of the Lagrangian ω instead of k. Henceforth we let k' represent k or $\sigma'_y + \omega$ and C represent the total coefficient on σ^2_g in either equation. With these identifications, the same model shows the present value of the certainty equivalents of the stochastic stream where the certainty equivalents are determined *either* by market equivalences *or* by investors' risk-averse utility functions directly. The relevant size of σ^2_g is determined by the stochastic process in profit rates given above.

The decision rule to optimize retentions and growth rates, for any fixed σ^2_p, is now given by the equalities in

$$\text{(14a)} \qquad \frac{\partial p_0}{\partial r} = p_0\left[-\frac{1}{x} + \frac{\dfrac{d\hat{g}}{dr} - 2arC\sigma^2_p}{k' - (\hat{g} - C\sigma^2_g)}\right] \geqslant 0$$

as $\rho_A - 2arC\sigma^2_p \geqslant \dfrac{k' - (\hat{g} - C\sigma^2_g)}{x}$ or as $\rho_A \geqslant y_e + 2arC\sigma^2_p$.

Model II: $a = 0$. Now assume for the moment that variances of growth rates are independent of retentions and expected growth rates. The decision rule with uncertainty introduced under this assumption is the same as under certainty in Model I (Section II) —i.e., make all investments whose expected rate of return $\geqslant$ the current earnings yield at market prices. Moreover, the general form of the function relating earnings yield to retained earnings will also be the same as in the certainty case — falling at a generally diminishing rate and then rising at a necessarily increasing rate — and the optimal investment and retention rates also minimize market earnings yields as before. The presence (and degree of) uncertainty clearly raises the cost of capital, and reduces both capital budgets and retentions: For any fixed k', r (or x) and $\hat{p}$, y_e will be greater (since P_0 will be lower) the larger is σ^2_p or C. It must be emphasized, however, that the uncertainty in this model raises the level of the whole y_e function in a compound fashion. It not only raises the intercept of the y_e curve on the vertical axis (when $r = 0$) to $k' + C\sigma^2_p$ (instead of k as in Model I); the uncertainty *also* makes its decline less rapid, raises and shifts its minimum point to the left, and makes the rising portion steeper — in effect, curling the whole y_e function upwards and to the left. All these points are illustrated in Figure I by the y_{e2} curve, which except for $\sigma^2_p > 0$ is drawn for the same parameter values as the y_{e1} certainty-model curve.

Model III: $a > 0$. Now let the variance of growth rates rise with the size of the budget, where $\sigma_g^2 = \sigma_p^2(1 + a\,r^2)$. With uncertainty admitted in this somewhat more general way, the earnings yield is (for any fixed k', x and p) *still higher* in Model III than in Model II. Although y_e still falls from $k' + C\sigma_p^2$ (when $r = 0$) at diminishing rate and then rises, the uncertainty now makes the decline in the y_e function still less rapid than in Model II, raises and shifts its minimum point still farther to the left, and makes the rising portion still steeper — in effect, *curling the whole y_e function still farther upwards and to the left* because σ_g^2 (with fixed σ_p^2) now increases quadratically with $f = r$.[3] This additional upward-leftward displacement in the y_e curve is clearly greater the greater the profit variance σ_p^2 and the larger the response fraction a. These three compounding effects of $C\sigma_p^2$ and a on the position and shape of the y_e function are illustrated in the y_{e3} curve in Figure I.

Optimum capital budgets, growth and retentions minimize earnings yields y_e on the stock in this model as in the previous ones — after all, min y_e with given *current* earnings is directly implied by max P_0 which is the object of the game. These three upward and leftward displacements in the location of the minimum earnings yield consequently mean that the *optimal retention ratio and growth rate both vary inversely and in compounded degree* with the uncertainty in profit rates σ_p^2 and with C; and optimal dividend payout ratios, as well as the "shortfall" of *optimal* growth rates *below* maximum possible *expected* growth rates, both increase in the same compound way with greater uncertainty.

But while optimum magnitudes for budget size, growth and retentions reflect the three compounding effects so far considered, still another element enters into the choice of the proper *decision*

3. Although we have developed "certainty equivalent" models it should be noted that the earnings yields y_e are *as if* the discount rate under certainty k were raised by an "uncertainty premium" as Irving Fisher suggested, *The Theory of Interest* (New York: Macmillan, 1930). The penultimate term in (14a) is the same as in (5) if $k_* = k' + C\sigma_g^2$ is used. But even in Model II the proper cutoff under growth is still the earnings yield, not k_*; and in Model III it is earnings yield *plus an additional risk term*. Moreover, since σ_g^2 and $2arC\sigma_p^2$ both depend on $f = r$, it is very clear that *there is no single "uncertainty discount rate"* (in the market or elsewhere), for use in equations to determine the optimal scale of investment (and rate of retentions or dividends) in growth situations when $a > 0$, which is *independent* of these decision variables — and this is true even when no outside equity is considered, and most notably even when the firms in question never have and never will use debt! The same observation holds *a fortiori* in later models.

rule. With uncertainty admitted in this more general way, it is clear from (14a) that *the current earnings yield of the stock is no longer the proper cutoff point or "cost of capital" for decisions regarding retentions and capital budgets.* The marginal cost of capital — the *minimum* acceptable marginal *expected* rate of return — is *necessarily greater* than y_e by an amount $2arC\sigma_p^2$. The addition of this latter term means that *the whole cost of capital* function[4] is *displaced upward and leftward still further than the y_e curve* — and by amounts that increase linearly with the retention ratio r, and that are proportional to the profit rate variance σ_p^2 and the composite parameter C, as well as to the budget response fraction a. *The fact and degree of the uncertainty thus has a fourfold cumulative effect upon the marginal "cost of capital"* in this more general model, and optimal budget size and retentions are determined by the intersection of the marginal expected rate of return ρ_A with this cost of capital $mcc_3 = y_e + 2arC\sigma_p^2$. Moreover, while optimal decisions determined by this intersection do minimize earnings yields they *do not minimize the marginal cost of capital* (as in Models I and II) — and this is true even though C and σ_p^2 are expectationally constant over time. With this (still limited) uncertainty in the picture, the *minimum* marginal cost of capital lies above and to the left of the minimum current earnings yield. ρ_A intersects mcc_3 at a point which

4. It may be noted that the term $2arC\sigma_p^2$ which must be added to the earnings yield y_e in determining the minimum acceptable marginal expected rate of return — the cost of capital in our derivation — is equal to C times the *marginal* variance in the growth rate with increments of retentions. Some readers may consequently wonder why we did not define a new concept of "Marginal Net Risk-Adjusted Expected Return" equal to $(\rho_A - 2arC\sigma_p^2)$, since in *conjunction with this measurement* of net marginal return the earnings yield can still serve as the proper measure of the cost of capital (due allowance being made for the effect of risk on y_e) in this as in the earlier models. This course was not followed for the following reasons: (1) the text definition running in terms of marginal expected returns seems much more in keeping with common usage, and especially so in our context in which the variance of the profit rate σ_p^2 is some constant independent of size of budget and retentions; (2) while mathematically unambiguous, the alternative definition as stated "in English" is likely to be misleading since the marginal expected return is properly adjusted by (but only by) the *marginal* (and not the average or total) variance of the growth rate (not profit rate); (3) for this reason, and because the "basic profit risk" $\sigma_{p_0}^2$ does not enter into the adjustment, the concept is thus very explicitly only a (particular) "risk-adjusted" *rather than* a "risk free" or "certainty equivalent" concept (with which it is likely to be erroneously associated in readers' minds), and the more precise phrasing "marginal-growth-rate-variance adjusted marginal expected return" seems quite cumbersome; and (4) while this alternative formulation would "save" the earnings yield as the proper cost of capital in the models so far, these models are deficient on other grounds (see following text) and the convenient generalization equating earnings yields with cost of capital must be abandoned anyway when more adequate models are used, as shown below.

lies directly above the min y_e, and at this point mcc_3 is *rising* with a slope $2aC\sigma_\psi^2$. All these relationships are illustrated in Figure I.

General Observations. While the models examined so far have provided many useful insights into the problems of optimal corporate investment and its financing, there are serious questions concerning their adequacy and applicability. In particular, the models are clearly *inapplicable* whenever $k' + C\sigma_g^2 > \hat{g}$ — our version of the well-known "growth stock paradox" with uncertainty recognized explicitly.[5] But the more substantive reservation or objection is that they ignore what seems to be an important and omnipresent fact of life. In particular, it appears that models are needed which will incorporate the (to me, at least, highly plausible and persuasive) observation that, *viewed as of a given point in time* and *in the minds of investors and managements alike,* the variance $\sigma_{\psi t}^2$ of the profitability of new investments to be made at different times in the future will be a monotone *increasing function* of their *futurity.* I consequently proceed to build new models which will allow for this phenomenon. As well as having much greater apparent plausibility and realism in their assumptions, they are entirely free from the "growth-stock" restrictions plaguing the three previous models.

IV. Optimal (Expectationally) Steady Growth, Capital Budgets, Dividends and Retentions, When $\sigma_{\psi t}^2$ Increases With Futurity

Our new models are the same as Models II and III in all respects except the assumption regarding corporate profit expectations. Specificially, I now assume that investors' and managements' expectations at any time t_0, regarding the position and shape of the profit possibility function ψ_{tt} which will be available to them τ

5. In this connection, it is not adequate to suggest that, if there are any companies in the market for which the model is inapplicable in a "partial equilibrium" analysis such as we have been making, this merely means that the value assigned to the given (risk-free) discount rate k is too low — that in a general equilibrium analysis in which k also is a variable, the equilibrium value of k would necessarily be sufficiently large as to satisfy the condition of equations (10) or (12) for every company in the market. But with I.B.M.'s, Polaroid's and other growth stocks (in the eyes of investors in recent years) in the market, some $\hat{g}$'s have undoubtedly been (expectationally) on the order of .15–.25 or perhaps more, and the markets' determination of k has surely *not* gone anywhere near so high in recent years. I assume there is agreement that the riskless k has been somewhat lower than the highest rate on government bonds. The more substantive defense of the models on this score would be that the relevant discount rate is $k' = \omega + \sigma'_{tj}$, not the riskless k itself; and that even if $\omega \sim k$ we might expect $\sigma'_{tj} + C\sigma_g^2 + \omega > \hat{g}$ in most cases. Whether this is true or not would require much careful research.

periods in the future, are such that, for any fixed $f = F^*/Y^*$ maintained throughout, $E(\tilde{p}_t) = \hat{p}$ constant over time in association with variances σ_{p_τ} which are rising linearly over time. Formally, instead of the previous profit function ψ_1, I now assume $\psi_2[f, \hat{p}, \sigma_p^2(t)]$ is invariant over time when $\sigma_p^2(t)$ is a rising linear function over τ (the futurity of the expectation). Instead of the previous assumption $p_t = \hat{p} + \epsilon$ with ϵ timewise independent, I now assume $p_{(t+1)} = p_t + e$ where e is normally distributed with mean zero and $\sigma_p^2 > 0$ constant over time but with all covariances between different times zero. The profit rate p_T is thus (expectationally) a cumulated random walk whose expected value is constant over time but with variance (respecting any future τ) of $\sigma^2_{p_\tau} = \sigma^2_{p_0} + \tau\sigma^2_e$ where $\sigma^2_{p_0}$ is the variance of the distribution of profitability at the moment t_0. To simplify the notation, let $2\beta = \sigma_e^2/\sigma^2_{p_0}$ and work with $\sigma^2_{p_\tau} = \sigma^2_{p_0}(1 + 2\beta\tau)$.

This stochastic process in profit rates leads to different stock price models and decision rules depending upon whether the variance in the initial growth rate $\sigma^2_{g_0}$ and its rate of increase over time are dependent upon the size of budget f. Ruling out the latter dependence for the moment, the variance of the growth rate at time t can be written $\sigma^2_{g_t} = \sigma^2_{p_0}(1 + a\tau^2 + 2\beta\tau)$, which upon integration makes the variance of cumulated growth over a span τ into the future $\sigma^2_{\tau g_\tau} = \tau\sigma_{p_0}(1 + a\tau^2 + \beta\tau) = \tau\sigma^2_{g_0} + \beta\sigma^2_{p_0}\tau^2$. The case where $\beta > 0$ but $a = 0$ will be denoted Model IV, and the case where both a and β are > 0 will be termed as Model V, but since the size of a only affects the term $\sigma^2_{g_0} = \sigma^2_p(1 + a\tau^2)$ we can continue to treat them together for a time.

The certainty equivalent, $\check{D}_T$, of the random dividend receipt $\tilde{D}_T$ at time τ now becomes

$$\check{D}_T = D_0 e^{\tau[\hat{g} - C(\sigma^2_{g_0} + \beta\sigma^2_{p_0}\tau)]}. \tag{15}$$

The sharp and fundamental contrast between this function and that used previously must be emphasized. The simpler stochastic process used in Models II and III had the highly undesirable property of producing certainty equivalent receipts, $\check{D}$, that rose *exponentially and forever at constant rate* $\hat{g} - \sigma'_{ij} - C\sigma^2_g > 0$. The introduction of uncertainty in this earlier function meant merely that an expected growth rate of 20 per cent will be reduced to an equivalent of, say, 18 per cent which within *those* models would have been treated as if it continued into the hereafter; in these earlier models, *any* certainty-equivalent growth rate greater than zero implied a $\check{D}_t$ which would increase *at this same rate forever* and without limit. But the certainty-equivalent dividend $\check{D}_t$ in *this* model based on equation (15) has the essential property of rising (at ever *diminishing* rates) *to a maximum* and then *falling* at ever increasing rates *to a certainty-equivalent receipt of zero* in the indefinite future. Investors using this model are not acting as if in the later future *any single* stock would offer the entire universe with ribbons around it! The general behavior of $\check{D}_t/D_0$ is illustrated in Table III and Figure III below.

Using (12) and (14) the price of the stock will now be

$$\text{(16a)}\qquad P_0 = \int_0^\infty D_0 e^{-\tau[k' - \hat{g} + C(\sigma^2_{g_0} + \beta\sigma^2_{r_0}\tau)]}\, d\tau, \text{ or}$$

$$\text{(16b)}\qquad P_0 = \frac{D_0 e^{w^2/2}}{\sqrt{2\beta C\sigma^2_{r_0}}} \int_w^\infty e^{-z^2/2}\, dz = \frac{D_0\sqrt{2\pi}}{\sqrt{2\beta C\sigma^2_{r_0}}} e^{w^2/2} G(w)$$

$$= \frac{D_0 G(w)}{f(w)\sqrt{2\beta C\sigma^2_{r_0}}}$$

where $w = (k' - \hat{g} + \sigma^2_{g_0})/\sqrt{2\beta C\sigma^2_{r_0}}$ and $z = w + \tau\sqrt{2\beta C\sigma^2_{r_0}}$, while $G(w)$ is the area in the right tail of the standardized normal frequency distribution and $f(w)$ is the ordinate of this distribution at the point w. It is immediately apparent from (16) that *these models are completely free of* any taint of (or restriction involving) *the "growth stock paradox"*: stock prices are finite regardless of the relative sizes of k and $\hat{g}$, and in particular when $\hat{g} > k$ and even when $\hat{g} > k' + C\sigma^2_{g_0}$.

In deriving decision rules, it will be convenient to write (16) in the form $P_0 = D_0 A$ where A represents the integral of the exponential function, and to let $z = t(k' - \hat{g} + C\sigma^2_{g_0}) + \beta\sigma^2_{r_0} t^2$, the entire exponent, so that $A = \int_0^\infty e^{-z}\, dt$. Now

$$\text{(17a)}\qquad \frac{\partial P_0}{\partial r} = P_0\left[-\frac{1}{x} + \partial A/\partial r/A\right] \gtreqless 0 \text{ as } x\frac{\partial A}{\partial r}/A \gtreqless 1,$$

and if we let $B = \int_0^\infty te^{-Z}\,dt$, we have immediately upon differentiation under the integral that

(17b)
$$\frac{\partial P_0}{\partial r} \gtrless 0 \text{ as } x(\rho_{\hat{}} - 2arC\sigma_{\dot{}}^2)\, B/A = x(\rho_{\hat{}} - 2arC\sigma_{\dot{}}^2)\, T \gtrless 1,$$

where $T = B/A$, i.e., that

(17c)
$$\frac{\partial p_0}{\partial r} \gtrless 0 \textit{ if and only if } \rho_{\hat{}} \gtrless (xT)^{-1} + 2arC\sigma_{\dot{}}^2.$$

A Generalization in Terms of "Duration." By inspection of the mathematical definitions of A and B, it is apparent that the ratio B/A, which we denote T, is (with our use of continuous flows and discounting) precisely what Macauley meant by the "duration" in his classic study of interest rates[6] and what Hicks[7] defined as the "average period" of a capitalistic production process or income stream in a related context. It is simply the weighted average time (of the dividend stream in our case) when present values of the receipts are used as weights. What equations (17) tells us then is: *added retentions and growth increase the price of a stock if but only if* (*a*) *the dividend payout* (*b*) *over a time period equal to the "duration" of the dividend stream of* (*c*) *the marginal expected rate of return adjusted for the marginal variance in the growth rate is* $\gtrless$ *unity* — the right-hand side being unity because the differential retention is in the denominator of both $\rho = \frac{\partial \hat{g}}{\partial r}$ and $2arC\sigma_{\dot{}}^2 = \frac{\partial ar^2C\sigma_{\dot{}}^2}{\partial r}$.[8] *This proposition is valid in all the models introduced in this paper.*[9]

It should be also noted that in full generality ideal policy does *not* call for *either* maximizing or minimizing the duration of the income stream. The optimum duration is given by the solution of

6. Frederick R. Macauley, *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856* (New York: National Bureau of Economic Research, 1938).

7. J. R. Hicks, *Value and Capital* (Oxford: Clarendon Press, 1939, 1946).

8. If both sides are multiplied by dr = the (small) increment in retentions, the equation would read that the cumulated undiscounted incremental cash dividend (adjusted for marginal growth rate variance) to shareholders over a period equal to the "duration" must be $\gtrless$ the amount of the incremental retention.

9. As shown below, however, in connection with Model VI, if the time rate of increase in growth-rate uncertainty is a function of the size of budget r — instead of being independent of this decision as in Models IV and V — the adjustment for the marginal variance in the growth rate becomes more complicated.

the equalities in (17) for $T = (1 - r^*)^{-1}\ (\rho_{\lambda}^* - 2ar^*C\sigma_p^2)^{-1}$; while maximizing the duration itself [cf. A4] would require increasing retentions until $\rho_{\lambda} = 2arC\sigma_p^2$ (since $r = 1$ is inadmissible as a *permanent* policy). The optimal "duration" of the certainty equivalents of the dividend stream, like the best expected growth rate and the most preferred average expected profit rate, is *less* than the maximum obtainable.

It can also be shown that the *duration* of the dividend stream, for fixed values of other parameters, is a monotone decreasing function of σ_p^2, a, C and β. (All the previous models are special cases of the more general Model V, with the certainty Model I being simply the very special case where $\sigma_p^2 = 0$.) Letting a subscript on T_i refer to duration in the indicated model, we consequently have $T_5 < T_4 < T_2 < T_1$ and $T_5 < T_3 < T_2 < T_1$ [1] for any given (admissible) set of values of other parameters. Since the adjustment for the marginal growth-rate variance also increases with a, $C = \alpha(1 + f_i)c/2$, and $\sigma_{p_0}^2$, it follows from (17) that *optimal dividend payout ratios vary directly — and optimal retention ratios and growth rates vary inversely — with all these elements of uncertainty ceteris paribus*. For precisely the same reasons, it is clear that the right-hand side of (17) — which is *the marginal cost of capital* defined as the minimum marginal expected return on current investment required to justify any additions to retentions, size of capital budget, and expected growth — *varies directly with each of these elements of uncertainty*: σ_p^2, a, β, and the future price-dividend variance ratio c — as well as the investors' risk-aversion parameter α, and (at least when probability distributions are identical, as we are assuming), the relative importance of the stock f_i.

One further generalization based on "duration" is of central importance: the earnings yield y_e is equal to the reciprocal of the product of the dividend payout ratio and duration *if but only if* the variance of expectations is independent of futurity; if expectation-variance increases with futurity, the earnings yield is necessarily less than this product. Mathematically, and in full generality, if we let $(xT)^{-1} = \lambda y_e$, we have $\lambda = 1$ *if and only if* $\beta = 0$ (as in Models I, II, and III) [2] while $\lambda > 1$ for all $\beta > 0$ (as in Models

1. T_4 may be either greater or less than T_3 because a appears in the one and β in the other.

2. In other words, in the first three models, the earnings yield on the stock is equal to the reciprocal of xT_i — i.e., $y_{ei} = (xT_i)^{-1}$ for models $i = 1, 2, 3,$ — which again gives, for any x or r, the relation $y_{e1} < y_{e2} < y_{e3}$ as pointed out in discussing these models — and by (17) the marginal cost of capital rises correspondingly (quite apart from the additional increase due to $arC\sigma_p^2$ in Model III).

IV, V, and VI). *In consequence, the marginal cost of capital is always greater than the earnings yields* (and often by substantial margins) *whenever the uncertainty of expectations increases with futurity.* Moreover, this excess of marginal cost of capital over earnings yields, is not due to any tendency of such uncertainty to depress earnings yields. On the contrary, as would be expected, it is immediately apparent from equations (14) and (16) that, for fixed earnings, payout ratios and other parameters, P_0 varies inversely and consequently *the earnings yield varies directly with β.* In sum, the fact that uncertainty increases with futurity ($\beta > 0$) raises *both* earnings yields and the marginal cost of capital — and both increasingly with the size of β — and introduces a positive "spread" between them so that earnings yields necessarily understate the marginal cost of capital (even when $a = 0$). The essential *economic* reasons for these and other theorems to be developed can best be brought out by a further separate examination of these models.

Model IV: $\beta > 0$, $a = 0$. To isolate the effect of β, we shall first examine Model IV and compare it with Model II. The degree of uncertainty (variance) is *independent* of rates of retentions and growth ($a = 0$) in both cases, but in Model IV there is a linear increase (proportional to $\beta > 0$) in expectational uncertainty *over time* concerning growth rates, while in Model II $\beta = 0$ and expectational variances do not increase with futurity.

The essential economic reason for the excess of the marginal cost of capital over the earnings yield when $\beta > 0$ (even when $a = 0$) turns on the fact that in all models *mcc* always inherently involves the *derivative* of price with respect to retentions (cf. equations (17) while y_e per se does not. It is also true that, in *both* Models II and IV (as in all others), any differential in the retention ratio r which increases growth rates will lengthen the duration T of the resulting dividend stream, since such higher retention rates necessarily mean that *relatively* less of the total expected cash flow will fall in earlier years and *relatively* more in later years. In Model II, however, the "uncertainty discount" in the eyes of investors (measured by unity *less* the ratio $\check{D}_t/D_0e^{\hat{g}t}$) is an exponentially linear function of time, so that its effect is *equivalent to* a single upward adjustment in a discount rate held constant over time. Moreover, under *these* conditions, the weighted average over future time of this (constant) adjustment in "as-if" discount rates is invariant to changes in duration induced by changing r. But in Model IV, on the other hand, the uncertainty discount is an exponentially *quadratic* function of time, and the increase in T with any positive differential r now means that *not only the existing level of the weighted average over future time of uncertainty discounts* (which *is* reflected in y_e) but *also the increase in this average* percentage uncertainty discount (which is *not* reflected in the *level* of y_e) must also be allowed for in measuring the marginal cost of capital. It is this latter component [3] which makes $\lambda > 0$ whenever $\beta > 0$ and $\rho_{\Lambda} > 0$.

Moreover, the situation is further compounded: for any given $\beta > 0$, λ *also* increases with increasing r throughout the relevant range. The essential economic reason is that the *rate of increase* in duration (as well as the duration itself) is a monotone increasing function of the expected growth rate (i.e., $\partial^2 T/\partial \hat{g}^2 > 0$), so that the *change* in the weighted-average uncertainty discounts will increase directly with growth rates (and hence with r up to the point where any further differential in r will not increase the expected growth rate). A $\beta > 0$ consequently not only raises earnings yields and introduces a positive spread between earnings yields and the marginal cost of capital, *but also* makes the relative spread larger the higher the retention ratio (or the smaller the dividend payout) up to values far beyond [4] the optimal r.

We also, as in our other models, find that the slope of the y_e function on r depends inversely on the difference between marginal expected returns ρ and the marginal cost of capital [$mcc_4 = (xT_4)^{-1}$] but the level of the latter function varies directly with the size of $\beta > 0$. Consequently, the level of the y_e function on r is raised for all $r > 0$ by amounts that (both absolutely and relatively) vary directly with β, and for any given $\beta > 0$, by relative amounts that increase progressively [5] with r — the whole function

3. Since stock prices are finite in models IV–VI, there is some discount rate, constant over time, which will give the same price as equation (16) for given values of D_0 and $\hat{g}$. Call this rate $\bar{k}_u$, defined implicitly by $P_0 = D_0/(\bar{k}_u - \hat{g}) = D_0 A$, so that $\bar{k}_u - \hat{g} = A^{-1}$. As shown in [A4], differentiating each term partially with r, yields $\partial \bar{k}_u/\partial r = \rho_{\Lambda} - (\rho_{\Lambda} - 2arC\sigma_p^2)\lambda^{-1}$, so that $\lambda = (\rho_{\Lambda} - 2arC\sigma_p^2)/(\rho_{\Lambda} - \partial \bar{k}_u/\partial r)$. With $a = 0$ in model IV, λ is equal to the ratio of the marginal expected return to its excess over the marginal increase in the "as if" discount rate. It is also shown that $\partial \bar{k}_u/\partial r > 0$ *and* $\partial^2 \bar{k}_u/\partial r^2 > 0$ for all $\beta > 0$ in models IV–VI.

4. Specifically for all r associated with marginal expected returns $\rho_{\Lambda} > 2arC\sigma_p^2$. The theorem is valid for r far beyond its optimum values since the optimal r_0 involves $\rho = (xT)^{-1} + 2arC\sigma_{p_0}^2$, and ρ declines with r.

5. The relative increase in y_e with β is a positive function of r for all r and β, and this produces the shift in min y_e noted below. The corresponding

being generally curled upward and to the left about its (higher) intercept at $r = 0$. This movement in turn has the further consequence of shifting the minimum of the y_e function on r to the left (corresponding to smaller retentions) by amounts that vary directly with β. This corresponds to our earlier proposition that increased variances and expectational uncertainty reduce optimum retentions, budget size and expected growth rates, since the optima minimize earnings yields in this as in all our other models. Perhaps more surprisingly, the fact that $\lambda > 1$ and increases with r when $\beta > 0$ means that the minimum mcc_4 necessarily lies above and *to the left* of the minimum y_{e4}. The decision rule that $\rho_{\Lambda} = mcc_4 = \lambda y_e = (xT_4)^{-1}$ consequently involves the intersection of ρ_{Λ} with mcc_4 at a point where the latter is rising and is above the earnings yield — once again, in contrast to Models I and II, *the marginal cost of capital is not only always greater than earnings yields, but it is not minimized* even though $a = 0$ in Model IV.

Finally, variations in the ratio $\lambda = (xT)^{-1}/y_e$ ($= mcc/y_e$ in Model IV) with β require comment. For all profit functions and values of r for which $k' + C\sigma^2_{g_0} > \hat{g}$, the ratio λ varies directly with β — and this will be true whenever $1 < \lambda < 1.5708$. On the other hand, when profit opportunities are sufficiently rich *and* retentions are sufficiently large to make $\hat{g} > k' + C\sigma^2_{g_0}$, then λ will vary inversely with β — more rapid rates of increase of expectational variances with futurity will reduce the *relative* understatement of the marginal cost of capital given by earnings yields. But this can occur *only if* the understatement is already greater than 57 per cent. Since both the size of λ *and* its relative rate of increase with g in this region are increasing functions of the growth rate g, λ can be *very* large when this "partial reversing" phenomenon occurs.[6]

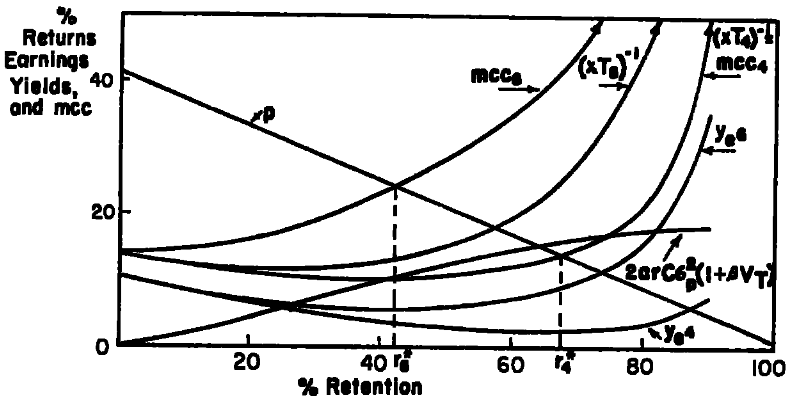
The effects of a $\beta > 0$ (specifically $\beta = .2$) are illustrated by the y_{e4} and mcc_4 curves in Figure I using the same values for all other variables as in the previous models. Since these illustra-

absolute increase is also necessarily positive so long as $\hat{g} < k' + C\sigma^2_g$ and consequently for small r values; and also for retention rates near optimal values (and for all larger r); but in strong growth situations it may be negative for (no more than) an intermediate range of r values.

6. Moreover, it is also of considerable interest to note that in full generality within this "upper region" where $\hat{g} > k' + C\sigma^2_{g_0}$, not only the *relative* increases in y_e with increasing β *but also the rate of increase with g in these relative increases* vary directly with the growth rate g. With relatively high growth rates, both earnings yields (and stock prices themselves) become extremely sensitive to the time-rate of increase in expectational variances measured by β.

FIGURE II

ILLUSTRATIVE MARGINAL EXPECTED RATES OF RETURN, EARNINGS YIELDS, AND MARGINAL COSTS OF CAPITAL: COMPARISONS OF MODELS IV AND VI.



Note: Data from Table II.

tions used a profit function sufficiently anaemic to avoid the growth stock paradox involved in earlier models, curves for Model IV are redrawn in Figure II with a profit function offering much richer (and for many companies, more realistic) growth potentials — specifically a (linear) average profit running from 41 per cent (at $r = 0$) to 21 per cent with all income retained — which would have been incompatible with earlier models. (β was also raised to .3 and $C\sigma^2_p$ was raised to .01).

Incidentally to avoid misunderstanding, these results and the model from which they are derived need to be clearly distinguished from a somewhat similar model advanced recently by Gordon.[7] Gordon simply *assumes* that the discount rate applicable to the future will be an increasing function of time[8] and asserts that *if* this assumption is true "the corporation's cost of capital is an increasing function of the rate of growth in its dividend."[9] In our Model IV the time-increase in the uncertainty discount is derived from explicit assumptions regarding the stochastic process of profit rates themselves *and* from the properties of an explicit utility function. Moreover, in Model IV the earnings yield *always declines* from $r = 0$ to the optimal retention rate, and the relevant cost of capital

7. *The Investment, Financing and Valuation of the Corporation, op. cit.*

8. Gordon used no utility function, and simply started with an assumption regarding variances of dividends (which is a variable to be derived); finding this assumption did *not* require that his k_t increase over time, he then assumed that k_t did so increase, and from *this* assumption asserted the conclusion stated in the text. Gordon's discount rate for a given future time k_t will be the *sum* of our ω (or the risk-free k) plus $C(\sigma^2_{\epsilon_0} + \beta\sigma^2_{p_0} t)$.

9. *Op. cit.*, p. 43.

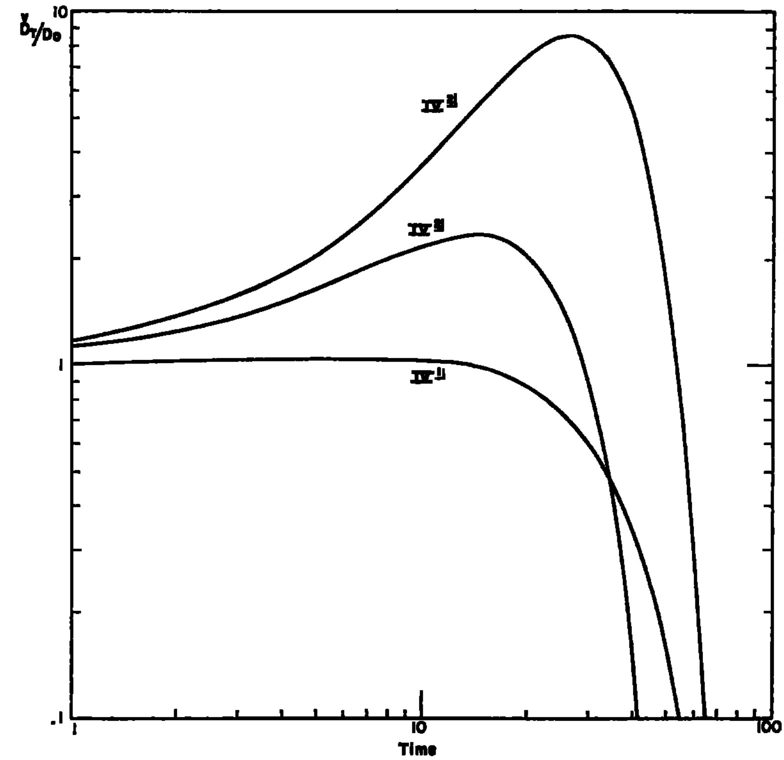
ILLUSTRATIVE TABLE II

ILLUSTRATIVE MARGINAL EXPECTED RATES OF RETURN, EARNINGS YIELDS, AND MARGINAL COSTS OF CAPITAL: COMPARISONS OF MODELS IV AND VI

	$\omega = .05$ $\sigma'_{ij} = .005$			$p = .41 - .2r$ $C\sigma^2_{p_0} = .010$			$a = 2$ $\beta = .3$			
r	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
p	.410	.390	.370	.350	.330	.310	.290	.270	.250	.230
g	0	.039	.074	.1050	.132	.155	.174	.189	.200	.207
ρ_A	.410	.370	.330	.290	.250	.210	.170	.130	.090	.050
y_{e4}	.1082	.0882	.0702	.0554	.0439	.0358	.0312	.0303	.0354	.0774
λ_4	1.281	1.417	1.639	1.965	2.440	3.1013	3.9584	4.964	5.974	6.923
$(xT_4)^{-1} = mcc_4$	.1386	.1250	.1151	.1090	.1071	.1110	.1235	.1504	.2115	.5363
$y_{a4} + g$	.1082	.1184	.1302	.1438	.1583	.1729	.1865	.1981	.2071	.2147
T_4 (years)	7.211	8.887	10.86	13.125	15.55	17.98	20.24	22.16	23.66	24.65
y_{e6}	.1082	.0889	.0739	.0641	.0599	.0622	.0728	.0984	.1587	.3589
λ_6	1.281	1.422	1.593	1.900	2.207	2.494	2.705	2.808	2.803	2.712
$(xT_6)^{-1}$	.1386	.1264	.1177	.1217	.1323	.1551	.1970	.2763	.4447	.9733
$2arC\sigma^2_p(1 + \beta V/T)$	0	.0212	.0439	.07475	.10278	.1283	.1500	.1672	.1801	.1893
mcc_6	.1386	.1476	.1616	.1965	.2351	.2834	.3470	.4435	.6248	1.0626
$y_{a6} + g$	.1082	.1190	.1131	.1499	.1679	.1861	.2031	.2185	.2317	.2429
T_6 (years)	7.211	8.787	10.615	11.733	12.597	12.89	12.68	12.09	11.24	10.27

FIGURE III

Illustrative Marginal Expected Rates of Return, Earnings Yields, and Marginal Costs of Capital: Comparisons of Models IV and VI.



[1] Based on profit function and other data in Table I.
[2] Based on profit function and other data in Table II.
Note: If plotted on the same figure, the certainty equivalents in Models I–III would rise exponentially.

mcc_4 (although greater and turning up sooner) will also *decline* for a range of r if the marginal return on investment ρ at $r = 0$ is large relative to the time-rate of increase in the variance of expectations $\beta C\sigma_p^2$ (as it will often be for companies worthy of being considered as "growth stocks") — and the range (and maximum extent) of falling mcc_4 will generally be greater the greater this ratio. These major differences in conclusions concerning the effects of an increasing uncertainty of expectations over the future arise from the fact that Gordon identifies the "cost of capital" with that single discount rate, constant over time, which if used in computing present values, would give the same present value as direct computation using some assumed pattern of increasing discount rates in each

TABLE III

CERTAINTY-EQUIVALENT CASH FLOW STREAMS IMPLIED BY DIFFERENT MODELS AT THEIR OPTIMUM RATES OF GROWTH

(Size at indicated time expressed as a ratio to D_0).

	Model					
	I[1]	II[1]	III[1]	IV[1]	IV[2]	VI[2]
r^*	.68	.46	.35	.31	.67	.417
$\hat{g}^*$	.0415	.0331	.02713	.02465	.1849	.1362
$\sigma_{ij} + C\sigma^2_{g^*}$	0	.0100	.0112	.01096	.02398	.0185
$\beta C\sigma^2_{g^*}\tau$	0	0	0	$.001\tau$	$.003\tau$	$.00404\tau$
Time[3]	$\hat{D}_\tau / D_0$ at Indicated Time					
1	1.042	1.023	1.016	1.014	1.171	1.120
5	1.231	1.179	1.083	1.044	2.074	1.628
10	1.514	1.392	1.173	1.037	3.703	2.166
20	2.293	1.939	1.375	0.887	7.523	2.092
35	4.274	3.185	1.745	0.471	7.075	0.436
50	7.965	5.286	2.218	0.162	1.742	0.015
100	63.43	10.074	4.919	$0.0^{3}18$	$0.0^{4}9$	$0.0^{10}4$
∞	∞	∞	∞	0	0	0
Max. value:	∞	∞	∞	1.048	8.53	2.36
Occurs at:				6.84	26.8	14.56

* Indicates optimizing value.
[1] Based on profit function and other data in Table I.
[2] Based on profit function and other data in Table II.
[3] Years in future.

period. This weighted average of increasing discount rates is, of course, a monotone increasing function of the growth rate of the dividend stream, but it is *not the "cost of capital" which is relevant for corporate decision-making purposes.*[1] As should be clear by now, the appropriate cost of capital to be compared with the marginal expected return on current investments is the *mcc* curve as derived and discussed above.

Model V: $\beta > 0$, $a > 0$. Model V differs from IV only by reintroducing a positive dependence between the variance of growth

1. This confusion also leads Gordon to regard the cost of capital as independent of the retention and growth rate when uncertainty and his time-discount rates are not increasing over time, whereas as shown in Models II and III admitting time-invariant uncertainty (and even in Model I under absolute certainty), the relevant cost of capital (to be compared with returns on current investments in growth situations) is *necessarily* a *decreasing* function of retentions and growth rates over a substantial range. These results stand in equally sharp contrast to Solomon's conclusions, as noted in my "The Cost of Capital and Optimal Financing of Corporate Growth," *op. cit.* Cf. Ezra Solomon, "Measuring a Company's Cost of Capital," *Journal of Business*, XXVIII (Oct. 1955), reprinted in Ezra Solomon (ed.), *The Management of Corporate Capital* (Glencoe, Ill.: The Free Press, 1959).

rates and relative size of budget (retentions in the present analysis). These models are related to each other when $\beta > 0$ as III and II were when $\beta = 0$. The effect of positive a's is once more to twist the y_e curve further upward and to the left (the intercept at $r = 0$ is unaffected but the rest of the curve is necessarily raised) and consequently to shift its minimum upward and further to the left (corresponding to still smaller r values). Optimal budget size, retentions and expected growth rates consequently vary inversely with a, *ceteris paribus*. Since these optima also vary inversely with β and $\sigma^2_{y_0}$, *ceteris paribus*, it follows that the restrictions induced by all the uncertainty factors are cumulative in their impact. Since these effects have been considered before, I shall not develop detailed decision rules for this intermediate model, but shall move directly to the final case which involves interaction effects.

Model VI: Full Interdependence. In Models IV and V the rate of increase in the expectational variance of growth rates with futurity was independent of the size of budget. We now drop this restrictive assumption, and let the expectational variance (viewed as of t_0) of the growth rate at some future $t_0 + \tau$ be $\sigma^2_{y_\tau} = \sigma^2_{y_0}(1 + ar^2)(1 + 2\beta\tau)$, which upon integration makes the variance of the cumulated growth over a span τ into the future $\sigma^2_{\tau_{g_\tau}} = \tau\sigma^2_{y_0}(1 + ar^2)(1 + \beta\tau) = \tau\sigma^2_{g_0} + \beta\sigma^2_{g_0}\tau^2$ with $\sigma^2_{g_0} = \sigma^2_{y_0}(1 + ar^2)$ as before. The certainty equivalent $\check{D}_\tau$ of the random receipt $\widetilde{D}_\tau$ now becomes

$$\check{D}_\tau = D_0 e^{\tau[\hat{g} - C\sigma^2_{g_0}(1+\beta\tau)]} \tag{18}$$

which is less than (15) for all $a > 0$; this certainty equivalent rises more slowly to a lower maximum (reached at an earlier time) and falls more rapidly than that used in Models IV and V. The price of the stock P_0 will still be given by equation (16) if $\sigma^2_{g_0}$ is substituted for $\sigma^2_{y_0}$ in the denominator, *and* in the specifications of w and z. With the corresponding substitution in the last term of Z, equation (17a) is also still valid, but we now have

$$\frac{\partial A}{\partial r} \Big/ A = (\rho_\Lambda - 2arC\sigma^2_{y_0})\, B/A - 2a\,\beta rC\sigma^2_{y_0}\, B^*/A$$
$$= (\rho_\Lambda - 2arC\sigma^2_{y_0})T - 2a\,\beta Cr\sigma^2_{y_0}\, V$$

where $B = \int_0^\infty te^{-Z}_e\, dt$ and $T = B/A$ as before, while

$B^* = \int_0^\infty t^2 e^{-Z}\, dt$ and $V = B^*/A$. The earlier decision rules in (17b) and (17c) now become

(19) $$\frac{\partial P_0}{\partial r} \gtreqless 0 \text{ as } x[(\rho - 2arC\sigma^2_{r_0})T - 2a\beta Cr\sigma^2_{r_0} V] \gtreqless 1,$$

or, letting $(xT_6)^{-1} = \lambda_6 y_{e6}$,

(20) $$\frac{\partial P_0}{\partial r} \gtreqless 0 \textit{ if and only if } \rho \gtreqless \lambda_6 y_{e6} + 2arC\sigma^2_{r_0}(1 + \beta V/T).$$

The generalization in terms of the Macauley-Hicks "duration" (p. 79 above) of the dividend stream still holds provided that marginal expected return is adjusted for the full marginal change in the variance of the growth rates which is now greater than in Models IV and V by virtue of the factor $\beta V/T > 0$. Just as T is the weighted average time or "duration" of the stream, with present values of payments as weights, V is the weighted mean squared time (or futurity from t_0) using the same weights.

While $y_{e6} = y_{e5}$ when retentions and growth are both zero, $y_{e6} > y_{e5}$ for all $r > 0$; y_{e6} is pivoted on the same intercept as y_{e5} on the vertical axis, but once again, y_{e6} is curled upward and leftward because of the additional interaction between a and β in Model VI. The minimum y_{e6} consequently lies above and to the left of min y_{e5}, and *the optimal rates of retention and growth ceteris paribus are consequently still lower than in Model V, and a fortiori lower than in all previous models. The "shortfall" of optimal levels of expected profit and growth rates below their respective maxima is correspondingly greater than in all previous models.* Moreover, the relevant cost of capital mcc_6 is correspondingly greater than in previous models: again due to the added interaction of a and β, $(xT_6)^{-1} = \lambda_6 y_{e6} > \lambda_5 y_{e5}$ by usually substantial margins for all retention ratios r; and the second compound term in (20) increases mcc_6 still further. While $2arC\sigma^2_p$ will be the same in Model VI as in V, the term $(1 + \beta V/T)$ in the present model is also an increasing function of retentions and growth rates so long as capital budgets are below optimum size. (The same relative shift in timing of receipts to the more distant future, which increases the mean duration T with higher growth rates, will increase the weighted mean square futurity V *relatively* even more.)

Because of all these several interacting and compounding effects of $\sigma^2_p > 0$, $a > 0$ and $\beta > 0$, with the rate of increase in expectational variance of growth rates allowed to be some increasing function of retentions and growth rates, the *minimum* acceptable marginal expectation of profit rates on incremental investments sufficient to justify the use of incremental funds to finance incremental investment — i.e., the marginal cost of capital — (a) is necessarily considerably greater than in any previous model, (b) is necessarily (and usually very substantially) *greater* than the earnings yield, and (c) is necessarily *rising* at the optimum point indicated by the intersection of ρ_A with mcc_6. Again these relationships hold in full generality [2] and are illustrated in Figure II. It will be apparent that throughout our analysis in all models, even though we have treated ρ, the marginal expected profit rate, as the demand-for-funds function, and *mcc* as a cost (or supply) of marginal funds function, the latter is inherently dependent on the former: while the position (height) of *mcc* at any point reflects the mean (or total) profit rate or profitability, growth, and variances, the *slope* of the *mcc* curve at any point *ceteris paribus* is greater negatively (or smaller positively) the greater the ρ_A at that scale of operations.[3]

Other Generalizations. Comparison of equations (20), (17) and (5a) reveals certain generalizations which are true in all six models. In all models, the price of the stock will be increased if and only if (a) the dividend payout of (b) the excess of the marginal expected returns on current investments over the *marginal* growth-rate-variance, *exceeds* (c) the product of λ and the dividend yield of the stock:

$$(21) \qquad \frac{\partial P_0}{\partial r} \gtrless 0 \Longleftrightarrow x[\rho_A - 2arC\gamma\sigma_p^2] \gtrless \lambda\, y_d$$

$$\text{where } \gamma = \begin{cases} 1 \text{ in models I–V} \\ (1 + \beta\, V/T) \text{ in model VI} \end{cases}$$

$$a = \begin{cases} 0 \text{ in models I, II, and IV} \\ a > 0 \text{ in models III, V, and VI} \end{cases}$$

$$\lambda \begin{cases} = 1 \text{ in models I–III} \\ > 1 \text{ in models IV–VI} \end{cases}$$

and where y_d is *ceteris paribus* an increasing function of a, β, $\sigma_{p_0}^2$, and $C = a(1 + f_t)c/2$ — and consequently of the investor's risk-

2. Mathematical proofs are given in [A5] of the mimeo. appendix.

3. It might also be noted that whereas the marginal cost of capital curve *necessarily* declines for a time from its value at $r = 0$ as r increases in Models I and II, and *usually* does so in Models III and IV, the condition for *mcc* to have a declining section for some range of $r > 0$ becomes increasingly stringent as we move through Models V and VI. Because of this pattern as one moves from model to model, two additional theorems are proved: First, that if in Models IV, V or VI $\partial mcc/\partial r = 0$ when $r = 0$, then $\partial P/\partial r > 0$ and $\rho > mcc$ so that positive retentions and expected growth rates are indicated. Correspondingly, proof is given that if $\partial P/\partial r \leqslant 0$ at $r = 0$, so that no retentions are justified, then $\partial mcc/\partial r > 0$: the marginal cost of capital *is necessarily rising* as it passes through the vertical axis. But for some ranges of positive slopes of *mcc* at the origin, positive retentions and expected growth rates will be optimal.

aversion parameter, the importance of the stock in their portfolio and the ratio of the variance of return due to resale price to the dividend-growth-rate-variance.

Moreover, for all rates of retention $r \leqslant r^*$ (the optimum), the dividend payout of the adjusted marginal expected return *plus* the expected growth rate will necessarily be *larger* than the *sum* of ω (or the riskfree rate k), the covariance term $\sigma_{\hat{y}}'$, and the variance of the growth rate $C\sigma^2_{g_0}$ at the time the decision is being made — except *only* at the optimum itself in Models I–III; the former sum will be larger than the latter even at the optimum in Models IV–VI. Even though the marginal cost of capital *may* be less than the discount rate adjusted upward for current risks, the investor's return from the expected growth rate and the dividend payout of marginal expected return (*after* adjustment for marginal growth-rate variance) will *never* fall short of this sum when optimizing policies are followed in *any* of these models. And, for those who wish to think in terms of a single discount rate, constant over time but reflecting the appropriate average of all allowances for uncertainties over the entire future, it may be noted that the sum of the dividend payout of the adjusted marginal expected return plus the expected growth rate will always *exceed this* "as-if" discount rate as well in Models IV–VI for all $r \leqslant r^*$, and will again be as low as this figure *only* at the optimizing point in Models I–III.

Finally, since the sum of the dividend yield and growth rate has been suggested by others as the relevant marginal cost of capital, it is worth noting that this identification is valid *only* at the origin in Models I–III, and that it *overstates* the required return throughout the range $0 < r \leqslant r^*$ in these models; that it *understates* the required returns at the origin in Models IV–VI and may or may not do so over the rest of the relevant range (both situations are illustrated in Table II) since the difference in the slopes of *mcc* and $(y_d + \hat{g})$ is a function of all other parameters. At the optimum for Model VI in Table II the proper marginal cost of capital is 24.3 per cent, while the sum of dividend yield and expected growth rate is 17.1 per cent — only about two-thirds as much.

V. Summary of Conclusions

1. This paper has examined the comparative stochastic dynamics of optimal corporate growth using the criterion of maximum present value of the common stock where the latter in turn is equal to the present value of the certainty equivalents of the pros-

pective cash flow (dividend) stream and the certainty equivalents are determined either by the indifference curves in purely competitive markets of optimizing investors *or* (in the absence of viable markets for the relevant "futures") directly by subjective risk-aversion (utility) considerations. The models had the property that *expected* values exhibit steady (exponential) growth, but certainty equivalents fall short of expected values by amounts which depend upon the particular stochastic process of future profit rates assumed, as well as market-equivalence or utility considerations. The later and more realistic models had the eminently desirable property that the *certainty equivalent of receipts* reach a well-defined *maximum and then progressively decline to zero.* For simplicity the analysis was confined to optimal internally financed growth and the appropriate *size* of capital budgets. To emphasize the relation of this analysis to current discussions in corporate finance, decision rules were derived in terms of an appropriately defined marginal cost of capital.

2. In order to encompass the essential characteristics of expectations of corporate growth continuing over substantial periods, we made the position of the marginal efficiency of capital function a monotone increasing function of a company's realized size (assets, capital stock, or earnings). When this is done the profit function relevant to any future period's investment is a function of amounts of investments in intervening time periods. After appropriately including the shift in such future profit opportunities attributable to this period's investment in the measure of *its* (average and marginal) profitability, we established that the *market discount rate is not* (except by coincidence) *the proper cutoff rate* for the marginal (expected) rate of return on current investment for establishing optimal target values for the relative size of capital budgets, amount of financing or ideal growth rates; nor is a discount rate reflecting current risks, nor even the single "as-if" discount rate assumed constant over time, the correct rate to use — and this is true even though, in all the models in this paper, the *sum* of the dividend payout of the marginal risk adjusted expected rate of return in the company *plus* the expected growth rate always equals or exceeds each of these criteria which have been suggested elsewhere.

3. The *current earnings yield* of the company's stock will provide the proper cutoff rate for (expectationally) continuing expansion financed by retained earnings *only if* there is complete certainty from here to eternity *or* if the uncertainty present is some constant profit-rate variance $\sigma^2_{r_0}$ which is *independent* of the size

of the capital budget and rate of investment *and* strictly *invariant over time.* Under any more general conditions, the proper marginal cost of capital (for comparison with the marginal expected rate of return on current investment) is *necessarily greater* than the current earnings yield on the stock — and by amounts which, both absolutely and relatively, vary directly with the relative size of the capital budget and its associated expected rate of growth. For growth rates at least equal to the sum of ω (or the riskless market discount rate) and growth-rate-variance, the difference is at least 57 per cent of the earnings yield and rises rapidly with greater growth rates up to and even beyond the optimum.

4. The current earnings yield itself is *not* some constant independent of the size of the capital budget as generally suggested; instead, it is a *falling* function of both the size of the budget and growth rates up to the optimal point of both. More generally, it falls at decreasing rates to the optimum budget size and thereafter rises at increasing rates. This is true in each of the six models analyzed in this paper ranging from pristine classical prescience through increasingly complex stochastic structures. Since the earnings yield does fall with increasing size of capital budgets between those yielding no expected growth and those of optimal size, the excess (noted in the previous point) of the relevant marginal cost of capital for retained earnings over the current earnings yield does not make the marginal cost of capital itself as great as would otherwise be the case. But *in full generality, the marginal cost of capital will be greater* (a) the greater the variance of profit rates, (b) the greater the ratio of future market-price variance to future earnings or dividend variance, and (c) the greater is the positive dependence, if any, of the variance of expected *growth* rates on relative size of capital budgets, (d) the greater the positive dependence of the variance of expectational *profit* rates on the futurity of the expectation — which will be present whenever expectational *profit* rates are regarded as a cumulative stochastic process with time-independent increments — and, if so, (e) the greater the positive dependence, if any, of time-rate-of-increase in expectational variances of future growth rates upon the relative size of capital budgets. With hyperbolic utility functions and lognormal probabilities, the investors' risk-aversion factor also raises the marginal cost of capital, since it enters directly into the relation between certainty-equivalents and expected values when these are derived from market-place equilibria. And, at least in the special case where investors' utility functions and probability distributions are identical, the

relative importance of the stock in portfolios also enters directly in determining risk discounts and certainty-equivalents.

5. For given degrees of uncertainty, higher expected average and marginal expected profit rates mean lower current earnings yields on the stock *ceteris paribus*. More significantly, higher marginal expected profit rates make the decline in the current earnings yield steeper (at all points short of the optimum). Moreover, in spite of the fact that making investments with high marginal returns increases expected future earnings, they *reduce* the marginal cost of capital in a partial equilibrium context — and do not increase it as standard writings aver.

6. Adherence to the criterion of maximizing the expected value of shareholders' equity does *not* imply maximizing expected growth rates as commonly assumed. This is true even under eternal certainty, and the "shortfall" between optimal and maximal expected growth with this criterion becomes absolutely and relatively larger as uncertainty is introduced, and does so in compounding fashion as described in point 4 above.

7. The marginal cost of capital in all models is an inverse function of the Hicks-Macauley "duration" of the implied income stream to investors. The optimal "duration" however, like optimal expected growth rates and the most preferred average expected profit rates, is *less* than the maximum obtainable.

8. When *any* allowance is introduced for the *variances* of expectations (of receipts or profit or growth rates) to increase with futurity, there is no possibility of a "growth-stock paradox" even in a partial equilibrium context.

9. Under generalized uncertainty (e.g., Models III–VI above), the relevant marginal cost of capital is not only greater than current earnings yields by amounts that increase with the size of the budget but is *necessarily rising* at the optimum point (where it intersects with marginal expected rate of return). Even though leverage per se has not yet been considered explicitly, it necessarily follows from the preceding analysis that *the conventional weighted-average-cost-of-capital rule is inherently erroneous and down-biased*. *Even if* a weighted average of equity and debt costs were the proper criterion, the average of earnings yield and interest cost would be *too low* because the relevant marginal cost of retained earnings is *greater* than the earnings yield (and the relevant marginal cost of outside equity still larger). If, for instance, both retained earnings and debt are to be used in financing, standard production theory insures that (a) the optimal *mix* will involve the

equalization of the two (interdependent) *marginal costs* and (b) the relevant marginal cost of (optimal-mix) finance for any sized budget will be *equal to the marginal costs of each type of finance used.*[4]

4. Cf. Lintner, "The Cost of Capital and Optimal Financing of Corporate Growth," *op. cit.*, and "Optimal Risk Bearing, Retentions and Leverage in Corporate Growth," *op. cit.*, for further development.

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